MATH-207(d) Analysis IV
Exercise session 9

Exercice 1. Let $\alpha \in \mathbb{R}$. Use the residue theorem to compute the following integral

$$\int_{-\infty}^{\infty} \frac{\sin(\alpha x)}{1+x^2} \mathrm{d}x. \tag{1}$$

You are *not* allowed to use the fact that the sine function is odd.

Exercice 2. Use the residue theorem to compute the Fourier transform $\hat{f}(\alpha)$ of the function

$$f(x) = \frac{x}{1+x^4}, \tag{2}$$

for all $\alpha \neq 0$.
Hint: The following fact might be useful:

$$z^2 = i \iff z = \pm\frac{1+i}{\sqrt{2}}, \qquad z^2 = -i \iff z = \pm\frac{1-i}{\sqrt{2}}.$$

Exercice 3. Let γ be a simple, closed, differentiable curve contained in the disk of radius 2 and centered at $z = 0$ in the complex plane. Use the residue theorem to calculate the following integral.

$$\int_\gamma \tan(z)\ \mathrm{d}z. \tag{3}$$

Exercice 4. Compute the following integral

$$\int_0^{2\pi} \frac{\cos(\theta)\sin(2\theta)}{5+3\cos(2\theta)}\ \mathrm{d}\theta. \tag{4}$$

Hint: A similar exercise was posed in the previous exercise sheet. As before, try to use the residue theorem by recasting this integral as a contour integral on the unit circle. The starting point is to observe that for $z = e^{i\theta}$ we have

$$\cos(\theta) = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right) \qquad \sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

Exercice 5. Calculate

$$\int_0^{2\pi} \frac{\sin^2(5\theta/2)}{\sin^2(\theta/2)}\ \mathrm{d}\theta. \tag{5}$$