MATH-207(d) Analysis IV
Exercise session 8

Exercice 1. Let $\gamma : [0,1] \to \mathbb{C}$ be a closed simple differentiable curve. Consider the functions

$$f(x) = e^{1/z}, \quad g(x) = e^{1/z^2}. \tag{1}$$

What are the possible values of the curve integrals of f and g over γ?

Exercice 2. Let $\alpha > 0$. Use the residue theorem to compute the following integrals

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{1+x^2}\mathrm{d}x, \qquad \int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{1+x^2}\mathrm{d}x. \tag{2}$$

Hint: for simplicity, you can first try $\alpha = 1$. You can easily express the second integral in terms of the first integral.

Exercice 3. Use the residue theorem to compute the integral

$$\int_{-\infty}^{\infty} \frac{e^{\alpha x i}}{4+x^4}\mathrm{d}x \tag{3}$$

Hint: The following facts might be useful:

$$z^2 = 2i \iff z = \pm(1+i), \qquad z^2 = -2i \iff z = \pm(1-i).$$

Exercice 4. Compute

$$\int_0^{2\pi} \frac{\cos^2\theta}{13 - 5\cos 2\theta}\mathrm{d}\theta. \tag{4}$$

Hint: Use the residue theorem by recasting this integral as a contour integral on the unit circle. The starting point is to observe that for $z = e^{i\theta}$ we have

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right)$$
$$\cos 2\theta = \frac{e^{2i\theta}+e^{-2i\theta}}{2} = \frac{1}{2}\left(z^2 + \frac{1}{z^2}\right).$$

Exercice 5. Let $\gamma = \left\{z \in \mathbb{C} : |z - \frac{\pi}{2}| = 1\right\}$. Compute the value of

$$\int_\gamma \frac{z^2 \sin(z)}{(z-\frac{\pi}{2})^2}\mathrm{d}z$$

with

(a) the Cauchy integral formula.

(b) the residue theorem.