MATH-207(d) Analysis IV
Exercise session 7

Exercice 1. Let $\gamma(t) = e^{it}$ be the standard parameterization of the unit circle. Compute the following integrals

$$A = \int_\gamma \frac{e^z}{z^2(z-2)}\, \mathrm{d}z, \qquad B = \int_\gamma \frac{\sin(z)}{z(z+2i)}\, \mathrm{d}z,$$
$$C = \int_\gamma \frac{z^3 - iz}{z(z-2i)}\, \mathrm{d}z, \qquad D = \int_\gamma \frac{1}{z} - \frac{1}{z^3} + z\sin(z)e^x\, \mathrm{d}z.$$

Exercice 2. Let $\gamma(t) = 10e^{it}$ be a parameterization of the circle around 0 with radius 10. Consider the function

$$f(z) = \frac{\sin(z)}{(z+1)(z+2)(z+3)}. \tag{1}$$

What are the singularities of f and what the orders of the poles? Calculate

$$\int_\gamma f(z)\mathrm{d}z. \tag{2}$$

Hint: express the numerator and denominator as Taylor series at different points; you don't need all the coefficients. To compute the residues, you can either use the formula from the lecture or the Cauchy integral formulas. Try both!

Exercice 3 (Essential singularities)**.** Consider the function $f(z) = e^{1/z}$.

(a) Determine the nature of the singularity.

(b) Let $z_0 = x_0 + iy_0$ be a point on the complex unit sphere. Study the value $f(tz_0)$ as t moves closer to zero depending on z_0. *Hint: in other words, we study how f behaves as we move straight towards the origin from different directions.*

(c) Given $y \in \mathbb{C}$, identify all the $w \in \mathbb{C}$ for which $y = e^w$.

(d) Show that in every small neighborhood of f, every complex number is attained by f infinitely often. Formally: show that for every $R > 0$ and $y \in \mathbb{C}$ there exist infinitely many $z \in B_R(0)$ for which $y = f(z)$.

Exercice 4. Consider the function $f(z) = \frac{1}{z^4-1}$.

(a) Determine the singularities of f and their nature.

(b) Let γ be a circle of radius $r > 0$ centered at the origin. Determine the values of the integral

$$\int_\gamma f(z)\, \mathrm{d}z \tag{3}$$

for cases $r = 0.5$, $r = 1$, and $r = 2$.

(c) More generally, determine the integral for any $r > 0$.

Exercice 5. Let $\gamma \subset \mathbb{C}$ be any simple close piecewise regular curve. Compute the following integrals depending on the curve γ.

(a) $\int_\gamma e^{1/z^2} \mathrm{d}z$

(b) $\int_\gamma \frac{z^2+2z+1}{(z-3)^3} \mathrm{d}z$

(c) $\int_\gamma \frac{e^{1/z}}{z^2} \mathrm{d}z$

(d) $\int_\gamma \frac{1}{(z-i)(z+2)^2(z-4)} \mathrm{d}z$

(e) $\int_\gamma \frac{\sin(z)}{z} \mathrm{d}z$

Exercice 6. Determine whether f has a singularity at $z_0 = 0$ and if yes, determine the order of the pole.

(a) $f(z) = \frac{z^2+2z+3}{z+1}$

(b) $f(z) = \frac{1+i}{z^2+z}$

(c) $f(z) = \frac{z^3}{z^2-z}$

(d) $f(z) = \frac{z^{-2}+z^{-1}+1}{z^{-2}+z+4z^2}$

(e) $f(z) = \frac{z^{-5}+z^{-2}+z^2}{z^{-2}+z+4z^2}$

(f) $f(z) = \frac{z^{-1}+1}{z^{-4}+3}$

(g) $f(z) = \frac{z^{-7}+1}{1+z}$