MATH-207(d) Analysis IV
Exercise session 5

Exercice 1. Compute the following integrals, where γ is a parameterization of the unit circle:

$$A = \int_\gamma \frac{\cos z}{z(z-2)}, \quad B = \int_\gamma \frac{\exp z^3 + z}{z(z+5)}, \quad C = \int_\gamma \frac{\exp z}{z(z+2i)}, \quad D = \int_\gamma \frac{\sin z}{z^4(z-3i)}.$$

Answer. Throughout this exercise, we apply the Cauchy integral formula. Specifically, we use the following variant:

$$\int_\gamma \frac{g(z)}{z^{n+1}}\, dz = \frac{2\pi i}{n!} g^{(n)}(0).$$

We proceed as follows.

(a) With $g(z) = \cos(z)/(z-2)$ and $n = 0$, we get

$$A = \int_\gamma \frac{g(z)}{z}\, dz = 2\pi i \cdot \frac{\cos(0)}{-2} = -\pi i.$$

(b) With $g(z) = (\exp z^3 + z)/(z+5)$ and $n = 0$, we get

$$B = \int_\gamma \frac{g(z)}{z}\, dz = 2\pi i \cdot \frac{\exp(0^3) + 0}{5} = \frac{2\pi i}{5}.$$

(c) With $g(z) = \exp(z)/(z+2i)$ and $n = 0$, we get

$$C = \int_\gamma \frac{g(z)}{z}\, dz = 2\pi i \cdot \frac{\exp(0)}{2i} = \pi.$$

(d) Finally, with $g(z) = \sin(z)/(z-3i)$ and $n = 3$, we get

$$D = \int_\gamma \frac{g(z)}{z^{3+1}}\, dz = \frac{2\pi i}{3!} g^{(3)}(0)..$$

We compute the third derivative at $z = 0$, using $\sin(0) = 0$ and $\cos(0) = 1$:

$$\begin{aligned}
g^{(3)}(z) &= \sin(z) \cdot \left(\frac{1}{z-3i}\right)''' + 3\sin'(z) \cdot \left(\frac{1}{z-3i}\right)'' + 3\sin''(z) \cdot \left(\frac{1}{z-3i}\right)' + \sin'''(z) \cdot \left(\frac{1}{z-3i}\right) \\
&= \sin(z) \cdot \left(\frac{1}{z-3i}\right)''' + 3\cos(z) \cdot \left(\frac{1}{z-3i}\right)'' - 3\sin(z) \cdot \left(\frac{1}{z-3i}\right)' - \cos(z) \cdot \left(\frac{1}{z-3i}\right) \\
&= 3\cos(z) \cdot \left(\frac{1}{z-3i}\right)'' - \cos(z) \cdot \left(\frac{1}{z-3i}\right) \\
&= 3\left(\frac{1}{z-3i}\right)'' - \left(\frac{1}{z-3i}\right).
\end{aligned}$$

Finally,

$$3\left(\frac{1}{z-3i}\right)'' - \left(\frac{1}{z-3i}\right) = 3(-1)(-2)\left(\frac{1}{z-3i}\right)^3 - \left(\frac{1}{z-3i}\right) = 3(-1)(-2)\frac{1}{-27i^3} - \frac{1}{-3i} = \frac{2}{-9i^3} +$$

Putting this all together, one finds:

$$D = \frac{2\pi i}{3!}\left(\frac{2}{9i} + \frac{1}{3i}\right) = \frac{2\pi}{3!}\frac{5}{9} = \frac{5}{27}\pi.$$

■

Exercice 2. Consider $f(z) = \log(1+z)$.

(a) Determine the largest region of $\mathbb{C}$ in which f is holomorphic.

(b) Compute the Taylor series in $z_0 = 0$ and determine the radius of convergence.

(c) Compute the Taylor series in $z_0 = i$ and determine the radius of convergence.

Answer.

(a) Since the function $\log y$ is holomorphic on

$$\mathbb{C} \setminus \{y \in \mathbb{C} : \operatorname{Im} y = 0, \operatorname{Re} y \leq 0\}$$

for $y = 1 + z$ we conclude that $\log(1+z)$ is holomorphic on

$$D = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im} z = 0, \operatorname{Re} z \leq -1\}.$$

(b) Since f is holomorphic at $z_0 = 0$, the Laurent series is the Taylor series, which reads

$$f(z) = \log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} z^n \text{ at } z_0 = 0.$$

One can recompute it by evaluating the derivatives f in $z_0 = 0$ and using the Taylor series formula

$$\sum_{n=0}^{+\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

The Taylor series in $z_0 = 0$ is convergent over the largest open disk around z_0 over which f is convergent. That disk has radius 1, because $z = -1$ there is a singularity. Hence the radius of convergence is $R = 1$.

We can also verify this directly:

$$|z| \leq \lim_{n\to\infty} \left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-1)^{n+1}}{n}}{\frac{(-1)^{n+2}}{n+1}}\right| = \lim_{n\to\infty} \frac{n+1}{n} = 1$$

(c) For $z_0 = i$, we compute the derivatives of f. We have

$$f'(z) = 1/(1+z), \quad f''(z) = -1/(1+z)^2, \quad f'''(z) = 2/(1+z)^3.$$

And in general

$$f^{(n)}(z) = (-1)^{n+1}\frac{(n-1)!}{(1+z)^n}.$$

Thus we get

$$f^{(n)}(i) = (-1)^{n+1}\frac{(n-1)!}{(1+i)^n}.$$

from which

$$f(z) = \log(1+i) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(1+i)^n}(z-i)^n.$$

The Taylor series in $z_0 = i$ is convergent over the largest open disk around z_0 over which f is convergent. That disk has radius $|i+1| = \sqrt{2}$, because thats the distance from i to the singularity at $z = -1$. Hence the radius of convergence is $R = \sqrt{2}$.

One can also verify that directly:

$$|z-i| \leq \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-1)^{n+1}}{n(1+i)^n}}{\frac{(-1)^{n+2}}{(n+1)(1+i)^{n+1}}}\right|$$
$$= \lim_{n\to\infty}|(1+i)|\frac{n+1}{n} = \sqrt{1^2+1^2} = \sqrt{2}.$$

■

Exercice 3. Find the coefficients of the Taylor series of the following functions around the specified point.

(a) $f(z) = e^z$ and $z_0 = 2$

(b) $f(z) = e^z$ and $z_0 = \pi\imath/2$

(c) $f(z) = e^{z^2}$ and $z_0 = 0$

(d) $f(z) = z^3$ and $z_0 = 1$

(e) $f(z) = \cos(z-3)$ and $z_0 = 3$

(f) $f(z) = \sin(z)^2$ and $z_0 = 0$

Answer. Let us first recall the definition of the Taylor series for smooth functions of a real variable. Let U be an open subset of $\mathbb{R}$, and let $f\colon U \subset \mathbb{R} \to \mathbb{R}$ be smooth. Then the Taylor series of f at the point $c \in U$ is the infinite series

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(a)}{n!}(x-c)^n,$$

where $f^{(n)}(a)$ is the n^{th} derivative of the function f evaluated at $x = a$.
The Taylor series for holomorphic functions of a complex variable are defined in the same manner. Indeed, let Ω be an open subset of $\mathbb{C}$, and let $f\colon \Omega \subset \mathbb{C} \to \mathbb{C}$ be holomorphic. Then the Taylor series of f at the point $z_0 \in \Omega$ is the infinite series

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n, \tag{1}$$

where $f^{(n)}(z_0)$ is the n^{th} complex derivative of the function f evaluated at $z = z_0$.

(a) Recall that the complex exponential function is holomorphic on the entire complex plane and we have that $f'(z) = \frac{d}{dz}\exp(z) = \exp(z)$ for all $z \in \mathbb{C}$. Using then Definition (1) of the Taylor series, we deduce that

$$a_n = \frac{f^{(n)}(2)}{n!} = \frac{e^2}{n!}.$$

(b) As before, we use the fact that the complex exponential function is holomorphic on the entire complex plane and $f'(z) = \frac{d}{dz}\exp(z) = \exp(z)$ for all $z \in \mathbb{C}$. Consequently, Definition (1) of the Taylor series, yields in this case that

$$a_n = \frac{f^{(n)}(2)}{n!} = \frac{e^{\pi\imath/2}}{n!} = \frac{\cos(\pi/2) + \imath\sin(\pi/2)}{n!} = \frac{\imath}{n!}.$$

(c) As the composition of two holomorphic functions, namely, the exponential function and the quadratic polynomial, the function $f(z) = e^{z^2}$ is clearly holomorphic on the entire complex plane. Unfortunately, a direct computation of the higher order complex derivatives of f is tedious, so we adopt an indirect approach. We begin by observing that the Taylor series of the function $g(y) = \exp(y)$ at any $y_0 = 0$ is given by

$$g(y) = \sum_{n=0}^{\infty} \frac{1}{n!} y^n.$$

Therefore, to obtain the Taylor series of $f(z) = \exp(z^2)$ at $z_0 = 0$, we simply substitute $y = z^2$ and deduce that

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!}(z^2)^n = \sum_{n=0}^{\infty} \frac{1}{n!} z^{2n}.$$

Comparing with Definition 1 of the Taylor series, we see that the Taylor coefficients of f are given by

$$a_{2n} = \frac{1}{n!} \quad \text{and} \quad a_{2n+1} = 0 \quad \forall n \in \mathbb{N}.$$

Let us remark here that this subsitution is allowed precisely because $\exp(z)$ equals the Taylor series for all $z \in \mathbb{C}$.

(d) As a polynomial, $f(z) = z^3$ is holomorphic on the complex plane and we have

$$f'(z) = 3z^2; \quad f''(z) = 6z; \quad f'''(z) = 6; \quad f^{(n)}(z) = 0 \ \forall n \geq 4.$$

Comparing with Definition 1 of the Taylor series, we see that the Taylor coefficients of f are given by

$$a_0 = f(z_0) = 1; \quad a_1 = f'(z_0) = 3; \qquad a_2 = \frac{f''(z_0)}{2} = 3; \quad a_3 = \frac{f'''(z_0)}{6} = 1;$$
$$a_n = f^{(n)}(z_0) = 0 \quad \forall n \geq 4.$$

In particular,

$$f(z) = 1 + 3(z-1) + 3(z-1)^2 + (z-1)^3.$$

(e) The function $f(z) = \cos(z-3)$ is holomorphic on the complex plane and a simple calculation shows that

$$f^{(4n)}(z_0) = \cos(z_0 - 3); \quad f^{(4n+1)}(z_0) = -\sin(z_0 - 3)$$
$$f^{(4n+2)}(z_0) = -\cos(z_0 - 3); \quad f^{(4n+3)}(z_0) = -\sin(z_0) \qquad \forall n \in \mathbb{N}.$$

Consequently, at $z_0 = 3$, we have

$$f^{(4n)}(z_0) = 1; \quad f^{(4n+1)}(z_0) = 0; \quad f^{(4n+2)}(z_0) = -1; \quad f^{(4n+3)}(z_0) = 0 \qquad \forall n \in \mathbb{N}.$$

Comparing with Definition 1 of the Taylor series, we see that the Taylor coefficients of f are given by

$$a_{4n} = \frac{1}{(4n)!}; \quad a_{4n+1} = 0; \quad a_{4n+2} = \frac{-1}{(4n+2)!}; \quad a_{4n+3} = 0 \qquad \forall n \in \mathbb{N}.$$

This problem can also be solved by a substitution argument similar to the one carried out in Exercise (c): we use the Taylor series of $\cos(x)$ around $x_0 = 0$ and substitute $x = z - 3$.

(f) As the product of two holomorphic functions, we see that $f(z) = \sin(z)^2$ is holomorphic on the complex plane. Unfortunately, a direct computation of the complex derivatives of $\sin(z)^2$ is tedious so we once gain rely on an indirect approach. We first use the well-known double-angle identity to write

$$\sin^2(z) = \frac{1 - \cos(2z)}{2}.$$

It therefore suffices to compute the Taylor coefficients of $g(z) = \cos(2z)$ at $z_0 = 0$. To do so, we will use a substitution argument similar to one used before. Indeed, from the previous exercise, we can immediately deduce that the Taylor series of $h(y) = \cos(y-3)$ at $y_0 = 3$ is given by

$$h(y) = \sum_{n=0}^{\infty} \frac{1}{(4n)!}(y-3)^{4n} + \sum_{n=0}^{\infty} \frac{-1}{(4n+2)!}(y-3)^{4n+2}$$

or, alternatively, in more compact form

$$h(y) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(y-3)^{2n}.$$

Using now the substitution $y = 2z + 3$, we deduce that the Taylor series of $g(z) = \cos(2z)$ at $z_0 = 0$ is given by

$$g(z) = \sum_{n=0}^{\infty} \frac{2^{4n}}{(4n)!} z^{4n} + \sum_{n=0}^{\infty} \frac{-2^{4n+2}}{(4n+2)!} z^{4n+2}$$

or, alternatively, in more compact form

$$g(z) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n}}{(2n)!} z^{2n} = \sum_{n=0}^{\infty} \frac{(-4)^n}{2n!} z^{2n}.$$

Consequently, with Definition 1 of the Taylor series, we see that the Taylor coefficients of f are given by

$$\begin{aligned}
a_0 &= \frac{1}{2} - \frac{1}{2} = 0, \\
a_{4n} &= -\frac{2^{4n}}{2(4n)!} \qquad \forall n \geq 1, \\
a_{4n+1} &= 0 \qquad \forall n \geq 0 \\
a_{4n+2} &= \frac{2^{4n+2}}{2(4n+2)!} \qquad \forall n \geq 0 \\
a_{4n+3} &= 0 \qquad \forall n \geq 0
\end{aligned}$$

■

Exercice 4. Let γ be a simple regular closed curve whose interior contains $z_0 = 0$. For any integer $k \in \mathbb{Z}$, explicitly compute the integral

$$\int_\gamma z^k \mathrm{d}z.$$

Hint: use the extended Cauchy theorem to replace γ by a curve that is easier to handle.

Answer. According to the extended Cauchy theorem, we can replace γ by the standard parameterization of the unit circle: $\gamma(t) = e^{it}$. We distinguish two cases for k: either $k = -1$ or $k \neq -1$.
The case $k = -1$ was already discussed in the lecture. We compute

$$\int_\gamma z^{-1} dz = \int_0^{2\pi} e^{-it} \cdot ie^{it} dt = \int_0^{2\pi} i\, dt = 2\pi i.$$

The case $k \neq -1$ is discussed as follows. We compute

$$\int_\gamma z^k dz = \int_0^{2\pi} (e^{it})^k \cdot ie^{it} dt = \int_0^{2\pi} ie^{i(k+1)t} dt.$$

Evaluating the integral yields:

$$\frac{i}{i(k+1)} \left[e^{i(k+1)t}\right]_0^{2\pi} = \frac{1}{k+1} \left[e^{i(k+1)2\pi} - e^0\right] = 0.$$

Therefore, the computed integral is summarized as:

$$\int_\gamma z^k dz = \begin{cases} 2\pi i & \text{if } k = -1, \\ 0 & \text{if } k \neq -1. \end{cases}$$

We remark that the case $k \geq 0$ can also be handled via the Cauchy theorem. However, the case $k < -1$ needs extra attention. ■

Extra Exercise. Cauchy-Riemann Equations.

Exercice 5. Let $\mathcal{O} \subseteq \mathbb{C}$ be an open set. Recall that a function $f : \mathcal{O} \to \mathbb{C}$ is complex differentiable at $z_0 \in \mathcal{O}$ if the limit

$$f'(z_0) := \lim_{z\to 0} \frac{f(z) - f(z_0)}{z - z_0} \tag{2}$$

exists and is finite. A variant of the statement that we have seen in class is as follows.

Theorem: Suppose that $f : \mathcal{O} \to \mathbb{C}$ with $f(x + iy) = u(x, y) + v(x, y)i$ is a complex function and that $z_0 = x_0 + y_0 i$. Then the following are equivalent:

(a) f is complex differentiable at $z_0 \in \mathbb{C}$.

(b) $u, v : \mathbb{R}^2 \to \mathbb{R}$ are differentiable at z_0 and satisfy the Cauchy-Riemann equations

$$\partial_x u = \partial_y, \quad \partial_y u = -\partial_x v. \tag{3}$$

In either case,

$$f'(z_0) = \partial_x u(x_0, y_0) + \partial_x v(x_0, y_0)i = \partial_y v(x_0, y_0) - \partial_y u(x_0, y_0)i$$

Prove that result. You can proceed with the following steps:

(a) Define a function $g : \mathcal{O} \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ by setting

$$g(x, y) := (u(x, y), v(x, y)) \,.$$

Show that it is differentiable and study its Jacobian to prove that u, v are differentiable and satisfy the Cauchy-Riemann equations.

(b) Conversely, given differentiable $u, v : \mathcal{O} \subseteq \mathbb{R}^2 \to \mathbb{R}$ satisfying the Cauchy-Riemann equations, define g as above and prove that f is complex differentiable.

(c) Assuming that f is complex differentiable, use the definition of complex differentiability and the definition of partial derivatives to find expressions for f'.

Answer.

(a) We write the complex derivative of f at z_0 as $f'(z_0) = w_0 = a_0 + b_0 i$. By definition of the complex derivative,

$$w_0 = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{4}$$

Hence,

$$0 = \lim_{z\to z_0} \left(w_0 - \frac{f(z) - f(z_0)}{z - z_0} \right) \tag{5}$$

We take the absolute value on both sides, and use that the absolute value is continuous over the complex numbers:

$$0 = \left| \lim_{z \to z_0} \left(w_0 - \frac{f(z) - f(z_0)}{z - z_0} \right) \right| \tag{6}$$

$$= \lim_{z \to z_0} \left| w_0 - \frac{f(z) - f(z_0)}{z - z_0} \right| \tag{7}$$

$$= \lim_{z \to z_0} \left| \frac{f(z) - f(z_0) - w_0(z - z_0)}{z - z_0} \right| \tag{8}$$

$$= \lim_{h \to 0} \left| \frac{f(z_0 + h) - f(z_0) - w_0 \cdot h}{h} \right| \tag{9}$$

$$= \lim_{h \to 0} \frac{|f(z_0 + h) - f(z_0) - w_0 \cdot h|}{|h|}. \tag{10}$$

We switch to the real point-of-view and introduce the auxiliary function

$$g(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} \tag{11}$$

We now have

$$0 = \lim_{h \to 0} \frac{\left| g(x_0 + h_x, y_0 + h_y) - g(x_0, y_0) - \begin{pmatrix} a_0 & -b_0 \\ b_0 & a_0 \end{pmatrix} \cdot h \right|}{|h|}. \tag{12}$$

By definition, the function g is differentiable at (x_0, y_0), and its Jacobian $Dg(x_0, y_0)$ there is

$$Dg(x_0, y_0) = \begin{pmatrix} a_0 & -b_0 \\ b_0 & a_0 \end{pmatrix} \tag{13}$$

However, we already know that

$$Dg(x_0, y_0) = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} \tag{14}$$

It follows that

$$a_0 = \partial_x u = \partial_y v, \qquad b_0 = \partial_y u = -\partial_x v. \tag{15}$$

(b) Now assume that $u, v : \mathcal{O} \to \mathbb{R}$ are differentiable at (x_0, y_0) and satisfy the Cauchy-Riemann equations. Now, the function

$$g(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix}$$

is differentiable at (x_0, y_0) as well. From the definition of differentiability of multivariate functions it follows that

$$\lim_{h \to 0} \frac{|g(x + h_x, y + h_y) - g(x_0, y_0) - Dg(x_0, y_0) \cdot h|}{|h|} = 0$$

where $Dg(x_0, y_0)$ is the Jacobian matrix

$$Dg(x_0, y_0) = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix} = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}.$$

In the last step, we have used the Cauchy-Riemann equations. Switching to the complex point-of-view, we can identify the Jacobian matrix with the complex number

$$w_0 = \partial_x u(x_0, y_0) + \partial_x v(x_0, y_0) \cdot i = \partial_y u(x_0, y_0) + \partial_y v(x_0, y_0) \cdot i. \tag{16}$$

We let $z_0 = (x_0, y_0)$ and obtain the limit

$$0 = \lim_{h \to 0} \frac{|f(z_0 + h) - f(z_0) - w_0 \cdot h|}{|h|}. \tag{17}$$

Here, working the in the complex setting, the number h is now complex and the product $w_0 \cdot h$ is complex multiplication. Equivalently,

$$\begin{align}
0 &= \lim_{z \to 0} \frac{|f(z) - f(z_0) - w_0 \cdot (z - z_0)|}{|z - z_0|} \tag{18}\\
&= \lim_{z \to 0} \left| \frac{f(z) - f(z_0) - w_0 \cdot (z - z_0)}{z - z_0} \right| \tag{19}\\
&= \lim_{z \to 0} \left| \frac{f(z) - f(z_0)}{z - z_0} - w_0 \right| \tag{20}\\
&= \left| \lim_{z \to 0} \frac{f(z) - f(z_0)}{z - z_0} - w_0 \right|. \tag{21}
\end{align}$$

Here, we have used that $|a/b| = |a|/|b|$ for any complex numbers $a, b \in \mathbb{C}$ and the fact that $|\cdot|$ is continuous over the complex numbers. We conclude that

$$\lim_{z \to 0} \frac{f(z) - f(z_0)}{z - z_0} = w_0. \tag{22}$$

By definition, f is complex differetiable at z_0 with $f'(z_0) = w_0$.

(c) We have already found the desired formulas for $f'(z_0)$. However, the following computations are interesting as well.

Since f is complex differentiable at z_0, we can consider a family $z = x + y_0$ such that $x \to x_0$ and use the limit property:

$$\begin{align*}
f'(z_0) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}\\
&= \lim_{x \to x_0} \frac{f(x + y_0 i) - f(x_0 + y_0 i)}{x - x_0}\\
&= \lim_{x \to x_0} \left(\frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0} i \right)\\
&= \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0} i\\
&= \partial_x u(x_0, y_0) + \partial_x v(x_0, y_0) i.
\end{align*}$$

Since f is complex differentiable at z_0, we can consider a family $z = x_0 + y$ such that

$y \to y_0$ and use the limit property:

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \\ &= \lim_{y \to y_0} \frac{f(x_0 + yi) - f(x_0 + y_0 i)}{yi - y_0 i} \\ &= \lim_{y \to y_0} \left(\frac{u(x_0, y) - u(x_0, y_0)}{yi - y_0 i} + \frac{v(x_0, y) - v(x_0, y_0)}{yi - y_0 i} i \right) \\ &= \lim_{y \to y_0} \frac{u(x_0, y) - u(x_0, y_0)}{yi - y_0 i} + \lim_{y \to y_0} \frac{v(x_0, y) - v(x_0, y_0)}{yi - y_0 i} i \\ &= \lim_{y \to y_0} -\partial_y u(x_0, y_0) i + \partial_y v(x_0, y_0). \end{aligned}$$

From these equations, it follows that

$$\begin{aligned} f'(z_0) &= \partial_x u(x_0, y_0) + \partial_x v(x_0, y_0) i \\ &= \partial_y v(x_0, y_0) - \partial_y u(x_0, y_0) i. \end{aligned}$$

NB: Any function $g : \mathbb{R}^2 \to \mathbb{R}^2$ can be interpreted as a function $f : \mathbb{C} \to \mathbb{C}$. In addition to that, if g is differentiable with a Jacobian whose entries satisfy the Cauchy-Riemann equations, then this g also represents a complex differentiable function. ■