MATH-207(d) Analysis IV
Exercise session 5

Exercice 1. Compute the following integrals, where γ is a parameterization of the unit circle:

$$A = \int_\gamma \frac{\cos z}{z(z-2)}, \quad B = \int_\gamma \frac{\exp z^3 + z}{z(z+5)}, \quad C = \int_\gamma \frac{\exp z}{z(z+2i)}, \quad D = \int_\gamma \frac{\sin z}{z^4(z-3i)}.$$

Exercice 2. Consider $f(z) = \log(1+z)$.

(a) Determine the largest region of $\mathbb{C}$ in which f is holomorphic.

(b) Compute the Taylor series in $z_0 = 0$ and determine the radius of convergence.

(c) Compute the Taylor series in $z_0 = i$ and determine the radius of convergence.

Exercice 3. Find the coefficients of the Taylor series of the following functions around the specified point.

(a) $f(z) = e^z$ and $z_0 = 2$

(b) $f(z) = e^z$ and $z_0 = \pi\imath/2$

(c) $f(z) = e^{z^2}$ and $z_0 = 0$

(d) $f(z) = z^3$ and $z_0 = 1$

(e) $f(z) = \cos(z-3)$ and $z_0 = 3$

(f) $f(z) = \sin(z)^2$ and $z_0 = 0$

Exercice 4. Let γ be a simple regular closed curve whose interior contains $z_0 = 0$. For any integer $k \in \mathbb{Z}$, explicitly compute the integral

$$\int_\gamma z^k \mathrm{d}z.$$

Hint: use the extended Cauchy theorem to replace γ by a curve that is easier to handle.

Extra Exercise. Cauchy-Riemann Equations.

Exercice 5. Let $\mathcal{O} \subseteq \mathbb{C}$ be an open set. Recall that a function $f : \mathcal{O} \to \mathbb{C}$ is complex differentiable at $z_0 \in \mathcal{O}$ if the limit

$$f'(z_0) := \lim_{z \to 0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1}$$

exists and is finite. A variant of the statement that we have seen in class is as follows.

Theorem: Suppose that $f : \mathcal{O} \to \mathbb{C}$ with $f(x+iy) = u(x,y) + v(x,y)i$ is a complex function and that $z_0 = x_0 + y_0 i$. Then the following are equivalent:

(a) f is complex differentiable at $z_0 \in \mathbb{C}$.

(b) $u, v : \mathbb{R}^2 \to \mathbb{R}$ are differentiable at z_0 and satisfy the Cauchy-Riemann equations

$$\partial_x u = \partial_y, \quad \partial_y u = -\partial_x v. \tag{2}$$

In either case,

$$f'(z_0) = \partial_x u(x_0, y_0) + \partial_x v(x_0, y_0)i = \partial_y v(x_0, y_0) - \partial_y u(x_0, y_0)i$$

Prove that result. You can proceed with the following steps:

(a) Define a function $g : \mathcal{O} \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ by setting

$$g(x, y) := (u(x, y), v(x, y)).$$

Show that it is differentiable and study its Jacobian to prove that u, v are differentiable and satisfy the Cauchy-Riemann equations.

(b) Conversely, given differentiable $u, v : \mathcal{O} \subseteq \mathbb{R}^2 \to \mathbb{R}$ satisfying the Cauchy-Riemann equations, define g as above and prove that f is complex differentiable.

(c) Assuming that f is complex differentiable, use the definition of complex differentiability and the definition of partial derivatives to find expressions for f'.