MATH-207(d) Analysis IV
Exercise session 1

1. Complex numbers algebra. Compute the cartesian representation.

(a) $z_1 = (2+i)i$

(b) $z_2 = (3-i)(4+i)$

(c) $z_3 = i(1+i)(1-i)$

(d) $z_4 = (2-i)^3$

(e) $z_5 = \frac{1}{i}$

(f) $z_6 = \frac{5-i}{i}$

(g) $z_7 = \frac{4+i}{3+2i}$

(h) $z_8 = \frac{2-i}{1-i}$

(i) $z_9 = \frac{3-2i}{2-i}$

(j) $z_{10} = (1-4i)^{-2}$

2. Complex number powers. Let us consider sequences of complex numbers of the form $z_n = z_0^{n+1}$, for every $n \in \mathbb{N}$ and some $z_0 \in \mathbb{C}$. Describe (with pictures or words) the aspect of the sequence of points in the complex plane for the following choices of z_0.

(a) $z_0 = i$

(b) $z_0 = \rho_0 e^{i\theta_0}$ with $\rho_0 = 0.99$ and $\theta_0 = \frac{91\pi}{180}$.

(c) $z_0 = \rho_0 e^{i\theta_0}$ with $\rho_0 = 0.99$ and $\theta_0 = \frac{89\pi}{180}$.

(d) $z_0 = \rho_0 e^{i\theta_0}$ with $\rho_0 = 0.99$ and $\theta_0 = \frac{61\pi}{180}$.

(e) $z_0 = \rho_0 e^{i\theta_0}$ with $\rho_0 = 0.99$ and $\theta_0 = (3-\sqrt{5})\pi \simeq \frac{137.5078\pi}{180}$.
You may need the help of a computer for this one! See also https://en.wikipedia.org/wiki/Golden_angle.

3. Euler's formula.

(a) For $z \in \mathbb{R}$, explain by pictures why

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}, \tag{1}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}. \tag{2}$$

(b) For $\alpha, \beta \in \mathbb{R}$, using that $e^{i\alpha}e^{i\beta} = e^{i(\alpha+\beta)}$ show that

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta),$$
$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta).$$

(c) We now extend Equations (1) and (2) to arbitrary complex numbers $z \in \mathbb{C}$. Show that these equations lead to the formula

$$\forall z \in \mathbb{C}: \quad e^{iz} = \cos(z) + i\sin(z).$$