MATH-207(c) Analysis IV
Exercise session 13

Exercice 1. Consider the periodic function $f(x)$ with period $T > 0$ that satisfies:

$$f(x) = \begin{cases} -1 & \frac{T}{2} < x \leq T \\ 1 & 0 \leq x \leq \frac{T}{2}. \end{cases}$$

Compute the Fourier coefficients and the Fourier series of f.

Exercice 2. Let $z^2 + pz + q$ be a polynomial with real coefficients $p, q \in \mathbb{R}$ and roots $z_1, z_2 \in \mathbb{C}$. Use the assumption that $p, q \in \mathbb{R}$ to show the following:

- The imaginary parts of the roots satisfy $\Im z_1 = -\Im z_2$.
- If any of the roots is not real, then both roots are not real, and then their real parts satisfy $\Re z_1 = \Re z_2$.

Exercice 3. Solve the initial value problem

$$u''''(t) = e^{-t}, \quad t > 0,$$
$$u(0) = u'(0) = u''(0) = u'''(0) = 0.$$

Exercice 4. Let $p, q \in \mathbb{R}$. Find the Laplace transform of any solution of

$$y'' + py' + qy = f.$$

Use this approach to find the solution in the special case $f = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 2$ and with parameters $p = 0$ and $q = 1$.

Exercice 5. Consider the following modified heat equation over $\mathbb{R}$:

$$\begin{cases} \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} - cu, & x \in \mathbb{R}, t > 0 \\ u(x, 0) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

Here, $\kappa > 0$ and $c > 0$ are material parameters, and u_0 denotes the initial values at time $t = 0$. We find a solution for this differential equation.

(a) Write down the Fourier transform (in x) of this differential equation.

(b) Determine $\hat{u}$.

(c) Identify how $\hat{u}$ can be written as the product of two Fourier transforms.

(d) Find the inverse Fourier transform of $\hat{u}$.

Exercice 6 (Extra, not for the exam)**.** We sometimes need to solve integral equations of the form

$$u(t) = g(t) + \int_0^t k(t-s)u(s)ds \tag{2}$$

where k is the so-called *kernel* and u is the unknown function.

(a) Assume that g and k are differentiable. Rewrite (2) as a first-order differential equation with unknown u. You need to use the Leibniz integral rule. What is $u(0)$?

(b) Find the Laplace transform of $u(t)$.

(c) Directly apply the Laplace transform to (2) and verify that the result is the same.

(d) Explain what happens if $g(t) = 0$ at all times $t \geq 0$.

(e) Find the solution to the equation

$$u(t) = e^{-t} + \int_0^t u(s)ds \tag{3}$$

(f) Find the solution to the equation

$$u(t) = e^{-t} + \int_0^t e^{-(t-s)}u(s)ds \tag{4}$$

(g) Find the solution to the equation

$$u(t) = t + \int_0^t e^{-(t-s)}u(s)ds \tag{5}$$

Remark: the physical background of the convolutional integral term $\int_0^t e^{-(t-s)}u(t)ds$ *is that it models a memory effect: the solution at time t is influenced by its values at previous times* $u(s)$*. At time* t*, the influence of the value* $u(s)$*, with* $0 \leq s \leq t$*, is weighted by* $e^{-(t-s)}$*. Hence, the the influence of the preceding values declines over time, and the more recent values of* u *have influence than the values at the beginning.*