MATH-207(d) Analysis IV
Exercise session 12

Exercice 1. Compute the Laplace transform of the function $f : [0, \infty) \to \mathbb{R}$ such that

$$f(t) = \begin{cases} 0 & \text{for } 0 \leq t \leq a \\ 1 & \text{for } t > a \end{cases}. \tag{1}$$

Exercice 2. Compute the Laplace transforms of the functions

$$f(t) = e^{at} \cosh(bt), \qquad g(t) = e^{at} \cosh(bt), \tag{2}$$

where $a, b \in \mathbb{C}$.

Exercice 3. Solve the ordinary differential equation

$$y'(t) + ay(t) = f(t), \quad t > 0, \tag{3}$$
$$y(0) = 1 \tag{4}$$

where $f(t) = te^{-3t}$. You are allowed to use all Laplace transforms that have been computed in the lecture or in previous exercises.

Exercice 4. Solve the ordinary differential equation

$$y''(t) + y(t) + 2y'(t) = f(t), \quad t > 0, \tag{5}$$
$$y(0) = 1, \quad y'(0) = 1. \tag{6}$$

in the two cases

(a) $f(t) = 0$,

(b) $f(t) = e^{-3t}$.

You are allowed to use all Laplace transforms that have been computed in the lecture or in previous exercises.

Exercice 5. Find a solution to the ordinary differential equation

$$y'''(t) + y'(t) = te^{-t}, \tag{7}$$
$$y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 1 \tag{8}$$

via the Laplace transform. You are allowed to use all Laplace transforms that have been computed in the lecture or in previous exercises.

Hint: The following identity might be useful:

$$\frac{1}{z^3 + z} = \frac{1}{z} - \frac{z}{z^2 + 1}.$$

Exercice 6. Solve the initial value problem

$$u''''(t) = e^{-t}, \quad t > 0,$$
$$u(0) = u'(0) = u''(0) = u'''(0) = 0.$$

Exercice 7. Let $z^2 + pz + q$ be a polynomial with real coefficients $p, q \in \mathbb{R}$ and roots $z_1, z_2 \in \mathbb{C}$. Use the assumption that $p, q \in \mathbb{R}$ to show the following:

- The imaginary parts of the roots satisfy $\Im z_1 = -\Im z_2$.
- If any of the roots is not real, then both roots are not real, and then their real parts satisfy $\Re z_1 = \Re z_2$.

Exercice 8. Let $p, q \in \mathbb{R}$. Find the Laplace transform of any solution of

$$y'' + py' + qy = f.$$

Use this approach to find the solution in the special case $f = 0$ with initial conditions $y(0) = 0$ and $y'(0) = 2$ and with parameters $p = 0$ and $q = 1$.

Exercice 9 (Extra)**.** The n-moment of a function $f : [0, \infty) \to \mathbb{R}$ is defined as

$$\mu_n := \int_0^\infty t^n f(t) \, \mathrm{d}t, \tag{9}$$

provided that this integral converges. Show that if all n-th moments of f converge and

$$\sup_{n \in \mathbb{N}} \int_0^\infty t^n |f(t)| \, \mathrm{d}t = \nu < \infty,$$

then

$$\mathfrak{L}[f](z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \mu_n z^n. \tag{10}$$