MATH-207(d) Analysis IV
Exercise session 11

Exercice 1. Let $a, b, c \in \mathbb{R}$ with $c \neq 0$. Given the Fourier transform $\hat{f}$ of f, you are asked to find the Fourier transform of:

- $g(x) = f(x + a)$
- $g(x) = e^{-ibx} f(x + a)$
- $g(x) = f(x/c)$ with $c \neq 0$.

Exercice 2. Compute the Laplace transforms of the following functions

$$f(t) = e^{i\theta t}, \quad g(t) = \sin(\omega t), \quad \text{and} \quad h(t) = \cos(\omega t),$$

where $\theta \in \mathbb{R}$ and $\omega > 0$.

Exercice 3. Compute the Laplace transform of

$$g(t) = te^{-at}. \tag{1}$$

Exercice 4. Compute the Laplace transforms of

$$f_1(t) = e^{-\beta t} \cos(\gamma t), \quad f_2(t) = e^{-\beta t} \sin(\gamma t). \tag{2}$$

In the case $\beta = 0$, you the functions from the previous exercise.

Exercice 5. (a) Consider the functions $f, g : [0, \infty) \to \mathbb{C}$ with

$$f(t) = t, \quad g(t) = e^{-t}. \tag{3}$$

Compute the convolution $f \star g$. Determine its Laplace transform using the convolution formula, and verify the result using a diirect calculation.

Exercice 6. Given $f : [0, 1] \to \mathbb{R}$. Let $y : [0, 1] \to \mathbb{R}$ such that

$$y''(t) = f(t), \quad 0 < t < 1$$
$$y(0) = 0, \quad y(1) = 0$$

Determine if the following statements are true or false. Justify your answer.

(a) Let $F(z)$ and $Y(z)$ denote the Laplce transforms of f and y. We have

$$Y(z) = \frac{F(z)}{z^2} + \frac{y'(0)}{z^2}.$$

(b) $y(t) = \int_0^t f(s)(t - s)\mathrm{d}s + ty'(0)$.

(c) $y(t) = \int_0^t f(s)(t-s)\mathrm{d}s + t\int_0^1 f(s)(1-s)\mathrm{d}s$

Exercice 7. Consider the following system of differential equations

$$\begin{cases} x'(t) = 2x(t) - 3y(t), & t > 0 \\ y'(t) = y(t) - 2x(t), & t > 0 \\ x(0) = 8, y(0) = 3. \end{cases}$$

and denote $X(z) = \mathcal{L}(x)(z)$ and $Y(z) = \mathcal{L}(y)(z)$.

(a) Applying the Laplace transform to the equations we find that

$$\begin{aligned} (z-a)X(z) + bY(z) &= 8 \\ cX(z) + (z-d)Y(z) &= 3 \end{aligned}$$

for some $a, b, c, d \in \mathbb{R}$. Find their value.

(b) Solving the above linear system for $X(z)$ and $Y(z)$ we find

$$X(z) = \frac{e}{z+1} + \frac{f}{z-4}, \quad Y(z) = \frac{g}{z+1} + \frac{h}{z-4}.$$

for some $e, f, g, h \in \mathbb{R}$. Find their value.

(c) Conclude by finding an expression for $x(t)$ and $y(t)$.