MATH-207(d) Analysis IV
Exercise session 6

Exercice 1. For each of the following functions compute the Laurent series in the given z_0, determine its region of convergence, specify the nature of the singularity and report the residue.

(a) $f(z) = z\cos\left(\frac{1}{z}\right)$ in $z_0 = 0$.

(b) $f(z) = e^{1/z}\sin\left(\frac{1}{z}\right)$ in $z_0 = 0$

(c) $f(z) = \frac{e^z}{(z-1)^2}$ in $z_0 = 1$

(d) $f(z) = \frac{\sin z}{(z-\pi)^2}$ in $z_0 = \pi$.

(e) $f(z) = \frac{\sqrt{z}}{(z-1)^2}$ in $z_0 = 1$

Exercice 2. Compute at least the singular part of the Laurent series of the following functions determine its region of convergence, specify the nature of the singularity and report the residue.

(a) $f(z) = \frac{\sin z}{\sin(z^2)}$ in $z_0 = 0$.

(b) $f(z) = \frac{1}{\cos^2\left(\frac{\pi}{2}z\right)}$ in $z_0 = 1$

(c) $f(z) = \frac{\log(1+z)}{\sin(z^2)}$ in $z_0 = 0$

(d) $f(z) = \frac{\sin z}{z(e^z-1)}$ in $z_0 = 0$

Exercice 3. Consider $f(z) = \frac{\sin\left(z^2+1\right)}{\left(z^2+1\right)^2}$.

(a) Find all singularities of f and determine their nature.

(b) Compute the residue in each singularity.

(c) Determine the region of convergence of the Laurent series around each singularity.

Exercice 4. Find the coefficients of the Laurent series of the following functions around the specified point. Determine the nature of the singularity.

(a) $g(z) = \dfrac{e^z}{(z-2)^2}$ and $z_0 = 2$

(b) $g(z) = \dfrac{2z^3 + 5z^2 + z + \imath}{z + \imath}$ and $z_0 = -\imath$

(c) $g(z) = \dfrac{\cos\left((z-1)^2\right)}{(z-1)^3}$ at $z_0 = 1$

(d) $g(z) = \dfrac{1}{z(z-1)^2}$ and $z_0 = 1$

Exercice 5. Find the coefficients of the Laurent series of the following functions around the specified point. Determine the nature of the singularity.

(a) $h(z) = \sin\left(\frac{1}{z}\right)$ and $z_0 = 0$

(b) $h(z) = \sin\left((z-1)^{-1}\right)$ and $z_0 = 1$

(c) $h(z) = (z-2)^2 \cos\left((z-2)^{-4}\right)$ and $z_0 = 2$

(d) $h(z) = (z+\imath)^5 e^{\left((z+\imath)^{-2}\right)}$ and $z_0 = -\imath$

Exercice 6. (Extra) Prove Liouville's theorem: if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded, that is, we have $|f(z)| \leq M$ for some $M \geq 0$, then f is constant. The following steps might be helpful.

(a) Write f as a power series with coefficients given by the Cauchy integral formula.

(b) Express the coefficients as line integrals over a circle of radius $r > 0$. Simplify the expression.

(c) Estimate the magnitude of the coefficients.