MATH-207(d) Analysis IV
Exercise session 4

1. Integrals on circles.
Let $\gamma_1 := \{z \in \mathbb{C} : |z| = 1\}$ be the circle in the complex plane with center 0 and radius 1 and $\gamma_2 := \{z \in \mathbb{C} : |z-2| = 1\}$ the circle with center 2 and radius 1. Compute the value of the following integrals, some of which have been discussed in the lecture.

(a) $\int_{\gamma_1} \frac{1}{z} \mathrm{d}z$.

(b) $\int_{\gamma_1} \frac{1}{z^2} \mathrm{d}z$.

(c) $\int_{\gamma_2} \frac{1}{z} \mathrm{d}z$.

(d) $\int_{\gamma_2} \frac{1}{z^2} \mathrm{d}z$.

2. More integrals on circles
Compute the following integrals.

(a) $\int_{\gamma} \frac{e^{2z}}{z} \mathrm{d}z$ with $\gamma = \{z \in \mathbb{C} : |z| = 2\}$.

(b) $\int_{\gamma} \frac{z^3+2z^2+2}{z-2i} \mathrm{d}z$ with $\gamma = \left\{z \in \mathbb{C} : |z-2i| = \frac{1}{4}\right\}$.

(c) $\int_{\gamma} \frac{\sin\left(2z^2+3z+1\right)}{z-\pi} \mathrm{d}z$ with $\gamma = \{z \in \mathbb{C} : |z-\pi| = 1\}$.

(d) $\int_{\gamma} \frac{3z^2+2z+\sin(z+1)}{(z-2)^2} \mathrm{d}z$ with $\gamma = \{z \in \mathbb{C} : |z-2| = 1\}$.

(e) $\int_{\gamma} \frac{e^z}{z(z+2)} \mathrm{d}z$ with $\gamma = \{z \in \mathbb{C} : |z| = 1\}$.

3. Another integral on a closed curve.
Let γ be any simple, closed and piecewise regular curve. Discuss the value of

$$\int_{\gamma} \frac{5z^2-3z+2}{(z-1)^3} \mathrm{d}z$$

depending on the curve γ. You must distinguish the cases:

- The pole of the integrand lies within the region encircled by the curve
- The pole of the integrand lies outside of the region encircled by the curve
- The pole lies on the curve

4. Yet another integral.
Compute the integral

$$\int_\gamma \frac{e^{z^2}}{(z-1)^2\,(z^2+4)}\mathrm{d}z$$

in the following cases :

(a) γ is the circle centered in $z=1$ of radius 1.

(b) γ is the boundary of the rectangle $\left\{z \in \mathbb{C} : -\frac{1}{2} \leq \mathrm{Re}(z) \leq \frac{1}{2},\, 0 \leq \mathrm{Im}(z) \leq 4\right\}$.

(c) γ is the boundary of the rectangle $\{z \in \mathbb{C} : -2 \leq \mathrm{Re}(z) \leq 0,\, -1 \leq \mathrm{Im}(z) \leq 1\}$.

5. Difficult integrals made "easy".
Complex analysis can be a powerful tool to calculate complicated integrals, even if those integ7rals do not involve complex numbers at all! The goal of this exercise is to show that

$$\int_{-\infty}^{+\infty} e^{-x^2}\cos(2bx)\mathrm{d}x = \sqrt{\pi}e^{-b^2} \tag{1}$$

$$\int_{-\infty}^{+\infty} e^{-x^2}\sin(2bx)\mathrm{d}x = 0. \tag{2}$$

(a) Argue that $f(z)=e^{-z^2}$ is holomorphic on $\mathbb{C}$.

(b) Consider the path $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ shown in Figure 1

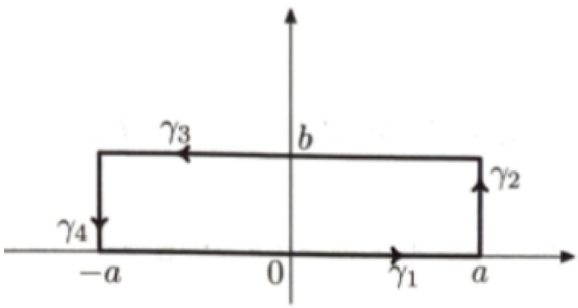

Figure 1: Path in complex plane for Exercise 4(b).

(i) Argue that $\int_\gamma f(z)\mathrm{d}z = 0$.

(ii) Show that $\lim\limits_{a\to+\infty}\int_{\gamma_2} f(z)\mathrm{d}z = \lim\limits_{a\to+\infty}\int_{\gamma_4} f(z)\mathrm{d}z = 0$.

(iii) Using that $\int_{-\infty}^{+\infty} e^{-x^2}\mathrm{d}x = \sqrt{\pi}$, conclude by showing (1) and (2).

6. Complex numbers and fluid dynamics.
Complex analysis has been a tool in fluid dynamics for a long time. Let $\mathcal{D} \subseteq \mathbb{R}^2$ be an open set. A vector field $\vec{F} : \mathcal{D} \to \mathbb{R}^2$ represents the velocity field of a fluid flow. The flow is called irrotational if $\mathrm{curl}\,\vec{F} = 0$ and incompressible if $\mathrm{div}\,\vec{F} = 0$ over $\mathcal{O}$.

(a) Show that the vector field

$$\vec{F} : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2, \qquad (x_1,x_2) \mapsto \left(\frac{x_1}{x_1^2+x_2^2}, \frac{x_2}{x_1^2+x_2^2}\right) \tag{3}$$

is irrotational and incompressible.

(b) Represent $\vec{F}$ by a complex function $f : \mathcal{O} \to \mathbb{C}$ for some open set $\mathcal{O} \subseteq \mathbb{C}$. Show that f is complex differentiable.

7. Contour integration.
Compute the following contour integrals.

(a) $\int_\gamma (z^2 + 1)\mathrm{d}z$ where $\gamma = [1, 1+i]$ (segment between 1 and $1+i$).

(b) $\int_\gamma \mathrm{Re}(z^2)\mathrm{d}z$, where $\gamma = \{z \in \mathbb{C} : |z| = 1\}$ (unit circle in 0).

Extra. Understanding complex numbers, once more If $z = x+iy$ is a complex number, what is the geometric interpretation of iz? More generally, for any $\theta \in \mathbb{R}$, what is the geometric interpretation of $e^{\theta i}z$? Finally, intepret this in terms of matrices.