MATH-207(d) Analysis IV
Exercise session 3

1. Holomorphic functions.
Show that the following functions are holomorphic on $\mathbb{C}$ and compute their derivative.

(a) $\cos(z) = \dfrac{e^{iz} + e^{-iz}}{2}$.

(b) $\cosh(z) = \dfrac{e^{z} + e^{-z}}{2}$.

(c) $\sinh(z) = \dfrac{e^{z} - e^{-z}}{2}$.

2. Holomorphic functions and harmonic maps.
Let $z = x + iy$, for $x, y \in \mathbb{R}$ and consider $f(z) = u(x, y) + iv(x, y)$, with $u, v \in C^2$. Show that if f is holomorphic on an open subset $\Omega \subseteq \mathbb{C}$, then u and v are harmonic functions, i.e.

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{and} \quad \Delta v = 0$$

3. More Holomorphic functions
Let $z = x + iy$, for $x, y \in \mathbb{R}$ and consider the function

$$f(z) = e^z = e^x e^{iy}.$$

Show that this function is holomorphic on $\mathbb{C}$ and compute its complex derivative.

4. Verifying Complex Differentiability
Express each of the following functions in the form

$$f(z) = u(x, y) + iv(x, y),$$

where u and v are real-valued functions and we represent any $z \in \mathbb{C}$ as $z = x + iy$ for some $x, y \in \mathbb{R}$.

Verify that the Cauchy-Riemann equations hold and compute the complex derivative.

(a) $f(z) = z^4$ over $\mathbb{C}$.

(b) $f(z) = e^{2z}$ over $\mathbb{C}$.

(c) $f(z) = 2z^2 + 1$ over $\mathbb{C}$.

(d) $f(z) = \dfrac{1}{z}$ over $\mathbb{C} \setminus \{0\}$.

(e) $f(z) = \dfrac{1}{z^2}$ over $\mathbb{C} \setminus \{0\}$.

5. The complex logarithm.
Let $z = x+iy$, for $x, y \in \mathbb{R}$ and denote $|z| = \sqrt{x^2+y^2}$ the modulus of z and $\arg z$ its argument. The (complex) logarithm of z is defined as

$$\log(z) = \log|z| + i \arg z, \tag{1}$$

where $-\pi < \arg z \leq \pi$ and $\log|z|$ is the natural logarithm of the real number $|z|$. Show that:

(a) The complex logarithm is well defined on $\mathbb{C} \setminus \{0\}$.

(b) The complex logarithm is holomorphic on

$$\mathcal{O} = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z \leq 0\}$$

and the derivative is $\frac{\mathrm{d}\log(z)}{\mathrm{d}z} = \frac{1}{z}$, $\forall z \in \mathcal{O}$. *You are only expected to show this when the real part is positive, that is,* $\Re z > 0$

6. The complex power.
Let $\gamma \in \mathbb{C}$ and define $f(z) = z^\gamma = e^{\gamma \log(z)}$.

(a) Show that in general f is holomorphic on $\mathcal{O} = \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im} z = 0 \text{ and } \operatorname{Re} z \leq 0\}$ and the derivative is $f'(z) = \gamma z^{\gamma-1}$.

(b) What can we say about the case $\gamma \in \mathbb{N}$?

7. Divergence and curl.
Suppose that $f(z) = u(x,y) + v(x,y)i$ is holomorphic. We define the vector field

$$F(x,y) = \begin{pmatrix} v \\ u \end{pmatrix}.$$

What is the divergence and the curl of F?