MATH-207(d) Analysis IV
Exercise session 2

1. Cartesian to polar representation. Find the absolute value of the following complex numbers in polar form, identifying both absolute value and principal argument:

$$z_1 = 3 + 4i, \qquad z_2 = -1 + i\sqrt{3}, \qquad z_3 = -2 - 2i, \qquad z_4 = 5i,$$
$$z_5 = -3, \qquad z_6 = 1 + i, \qquad z_7 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}, \qquad z_8 = -2i.$$

Hint: in computing the principal argument, first think of how you can obtain the angle from (x, y).

2. Polar to Cartesian representation. Find the Cartesian representation of the following complex numbers:

(a) $r = 2$, $\theta = \frac{\pi}{4}$

(b) $r = 3$, $\theta = -\frac{\pi}{3}$

(c) $r = 4$, $\theta = \pi$

(d) $r = 1$, $\theta = \frac{3\pi}{2}$

3. Representation of functions. Write $f : \mathbb{C} \to \mathbb{C}$ in the form $f = u + iv$, where u and v are its real and imaginary parts.

- $f(z) = z^3$
- $f(z) = \frac{1}{z+1}$
- $f(z) = e^{2z}$
- $f(z) = \sin z$
- $f(z) = \cos z$
- $f(z) = \sinh z$
- $f(z) = \cosh z$
- $f(z) = \frac{z}{z+1}$
- $f(z) = \frac{1}{z^2+1}$
- $f(z) = \frac{z^2}{z-i}$

Determine a formula for the absolute value in each case, simplifying as much as possible.

4. Double exponentials.
Determine the real and imaginary part of the following functions:

$$f(z) = e^{e^z}, \quad g(z) = \frac{1}{f(z)},$$

where $z = x + iy$. Describe the function f in the two special cases $x = 0$ and $y = 0$.

5. Matrix representation of complex numbers.
Given a complex number $z = x + yi$, we define a matrix

$$M(z) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}.$$

Show the following:

$$M(z_1 + z_2) = M(z_1) + M(z_2), \quad M(z_1 \cdot z_2) = M(z_1) \cdot M(z_2),$$
$$M(z)^{-1} = M\left(z^{-1}\right) \text{ if } z \neq 0.$$

6. Review Fourier series and Poisson problem.
Consider the boundary value problem of finding a function $u : [0, 1] \to \mathbb{R}$ such that

$$u''(x) = x, \qquad 0 < x < 1,$$
$$u(0) = u(1) = 0.$$

Express the solution as a Fourier series.