27.03.2025

Example: Let $f(z) = \dfrac{z}{\sin z}$ and $z_0 = 0$

$$f(z) = \frac{z}{z - z^3/3! + z^5/5! + \cdots} = \frac{1}{1 - z^2/3! + z^4/5! + \cdots}$$

$$f(z_0) = \lim_{z \to z_0} f(z) = 1/1 = 1$$

The function is holomorphic at $z_0 = 0$.

Example Let $f(z) = \dfrac{e^z - 1 - z}{z^2}$ and $z_0 = 0$

$$f(z) = \frac{\left(\sum_{n=0}^{\infty} z^n/n!\right) - 1 - z}{z^2} = \frac{z^2/2! + z^3/3! + z^4/4! + \cdots}{z^2}$$

$$= 1/2! + z/3! + z^2/4! + \cdots$$

The function is holomorphic, in particular,

$$f(z_0) = 1/2! = 1/2$$

Summary: if the numerator and denominator have the leading power in their Laurent series, then we factor out the common power $(z-z_0)^m$ and take the limit $z \to z_0$.

c) Suppose that we have Laurent series

$$p(z) = a_k (z-z_0)^k + a_{k+1}(z-z_0)^{k+1} + \dots \qquad a_k \neq 0$$

$$q(z) = b_\ell (z-z_0)^\ell + b_{\ell+1}(z-z_0)^{\ell+1} + \dots \qquad b_\ell \neq 0$$

Let $f(z) = p(z)/q(z)$

We have discussed the case $k = \ell$ already. Next, $k \neq \ell$.

$\boxed{\text{If } k \geq \ell}$, then f is holomorphic at z_0 and

$$f(z_0) = \lim_{z \to z_0} \frac{a_k (z - z_0)^k + \cdots}{b_\ell (z - z_0)^\ell + \cdots}$$

$$= \lim_{z \to z_0} \frac{a_k (z - z_0)^{k-\ell} + \cdots}{b_\ell + \cdots}$$

[reduce by common factor $(z - z_0)^\ell$]

Since $b_\ell \neq 0$, the limit goes to a finite value in $\mathbb{C}$, namely:

If $k = \ell$, then $f(z_0) = \dfrac{a_k}{b_\ell}$

If $k > \ell$, then $f(z_0) = 0$

$\boxed{\text{If } k < l}$, then z_0 is a pole of order $l-k$:

$$f(z) = \frac{a_k (z-z_0)^k + \dots}{b_l (z-z_0)^l + \dots}$$

$$= \frac{a_k}{b_l (z-z_0)^{l-k}} + \frac{a_{k+1}(z-z_0)}{b_l (z-z_0)^{l+1-k}} + \dots$$

Asymptotic argument: close to z_0, the higher powers of $z-z_0$ are small, and therefore,

$$f(z) \approx \frac{a_k}{b_l (z-z_0)^{l-k}} \quad \text{when } z \text{ is close to } z_0.$$

[Summary: this classifies the behavior of $f(z) = p(z)/q(z)$ close to $z_0 \in \mathbb{C}$ in terms of the Laurent series of p and q around z_0.]

Example: $f(z) = \dfrac{\sin(z)}{2z^3 + z^4}$ has a pole of order 2 at $z_0 = 0$

$$f(z) = \frac{z - z^3/3! + z^5/5! - \dots}{2z^3 + z^4}$$

$$= \frac{1 - z^2/3! + z^4/5! - \dots}{2z^2 + z^3}$$

Close to z_0, we have $f(z) \approx \frac{1}{2} z^{-2}$. We can already conclude

$$f(z) = c_{-2} z^{-2} + c_{-1} z^{-1} + c_0 + \dots$$

$\frac{1}{2}$

III. 4 Residues

Definition: The coefficient c_{-1} in the Laurent series of f at z_0 is called the residue of f at z_0:

$$\mathrm{Res}_{z_0}(f) := c_{-1} = \frac{1}{2\pi i}\int_\gamma f(z)\,dz$$

where γ is a small circle around z_0.

Example: If f is holomorphic over the interior of γ, then $c_{-1} = 0$.

Example: $\mathrm{Res}_0\left(\frac{1}{z}\right) = \frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz = \frac{2\pi i}{2\pi i} = 1$

Suppose $f(z)$ has a singularity of order m at z_0.

$$f(z) = c_{-m}(z-z_0)^{-m} + \dots + c_{-1}(z-z_0)^{-1} + c_0 + c_1(z-z_0) + \dots$$

Then

$m-1$ term in the Taylor series of g

$$g(z) := (z-z_0)^m f(z)$$

$$= c_{-m} + c_{-m+1}(z-z_0) + \dots + c_{-1}(z-z_0)^{m-1} + \dots$$

is the Taylor series of the holomorphic function $g(z)$ around z_0.

Using the well-known formula for the Taylor series coefficients, we obtain the coefficient c_{-1}

$$c_{-1} = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}} g(z_0) = \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left[(z-z_0)^m f(z)\right]$$

It is often convenient to write this as a limit :

$$c_{-1} = \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} g(z_0)$$

$$= \frac{1}{(m-1)!} \lim_{z \to z_0} g^{(m-1)}(z)$$

$$= \frac{1}{(m-1)!} \lim_{z \to z_0} \left[(z - z_0)^m f(z) \right]^{(m-1)}$$

This formula is helpful if we know the order of the pole.

<u>Example</u> Let $f(z) = 1/\sin(z)$ and $z_0 = 0$. Then

$$f(z) = \frac{1}{z - z^3/3! + \ldots}$$

Hence f has a pole of order 1. We compute the residue

$$\mathrm{Res}_0(f) = c_{-1} \text{ in the Laurent series of } f \text{ at } z_0$$

$$= \lim_{z \to 0} \left[z\, f(z) \right]$$

$$= \lim_{z \to 0} \frac{1}{1 - z^2/3! + z^4/5! - \dots} = 1$$

<u>Interlude</u>: throughout this class, we have used and will use closed curves only in counterclockwise orientation.

If we change the orientation, then the sign of the integral changes.

$$\circlearrowleft \;=\; -\circlearrowright$$

We will only use closed curves in counterclockwise direction, unless mentioned explicitly.

IV Residue theorem and its applications

IV. 1 Main result

If $f : \mathbb{C} \setminus \{z_0\} \to \mathbb{C}$ is a holomorphic function, and γ is a closed simple differentiable curve around z_0, then

$$\mathrm{Res}_{z_0}(f) = \frac{1}{2\pi i} \int_\gamma f(z)\, dz$$

That means, we can express the curve integral as

$$\int_\gamma f(z)\,dz = 2\pi i \cdot \operatorname{Res}_{z_0}(f)$$

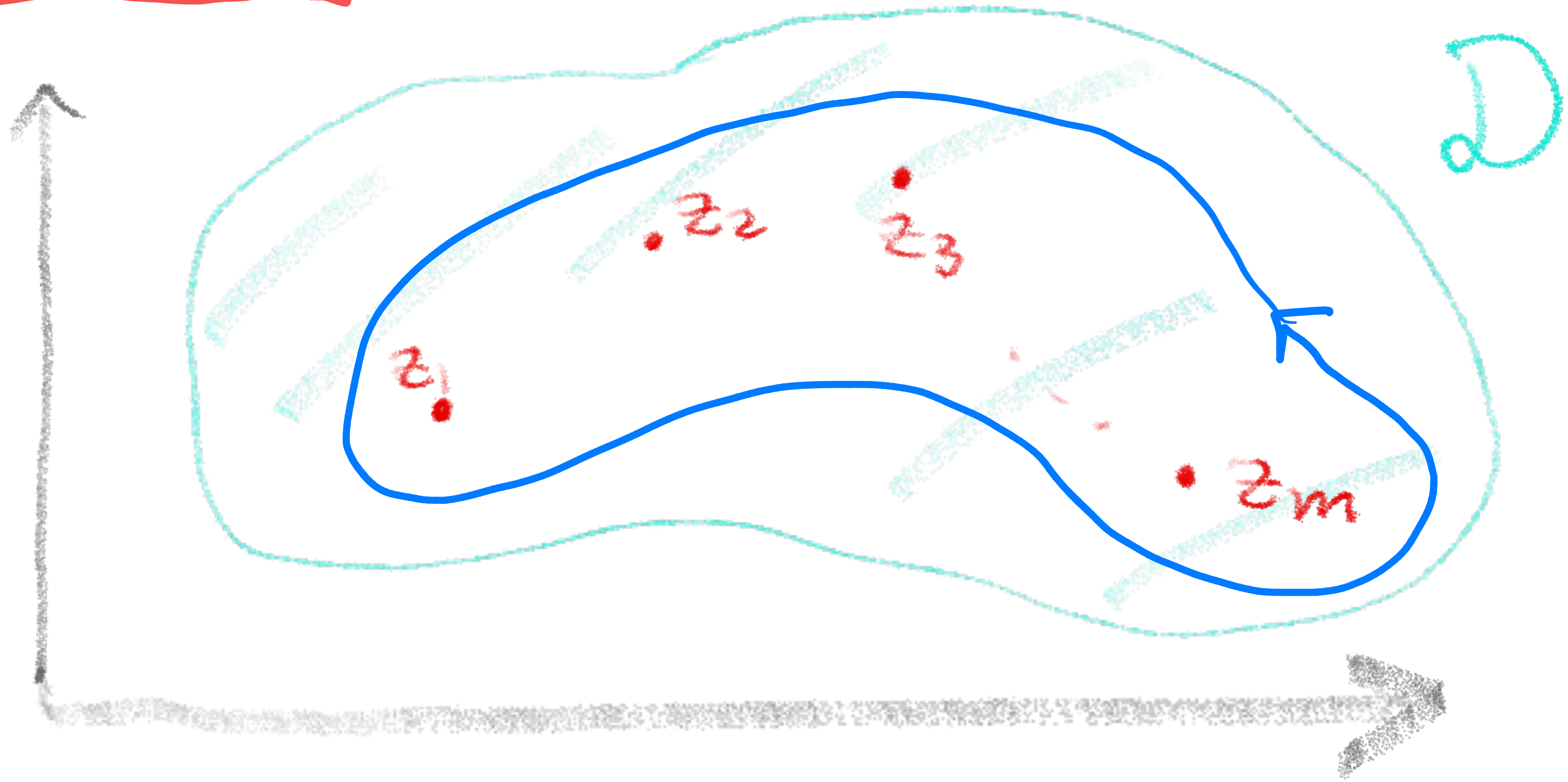
More generally, the residue theorem holds:

<u>Thm</u>: Suppose that $\mathcal{D} \subseteq \mathbb{C}$ is a simply-connected open set and let γ be a closed simple differentiable curve in $\mathcal{D}$ that encircles the points $z_1, z_2, z_3, \ldots, z_m \in \operatorname{int}\gamma$

If $f : \mathcal{D} \setminus \{z_1, z_2, \ldots, z_m\} \to \mathbb{C}$ is holomorphic, then

$$\int_\gamma f(z)\,dz = 2\pi i \left(\operatorname{Res}_{z_1}(f) + \ldots + \operatorname{Res}_{z_m}(f) \right)$$

Illustration



Remark: According to the residue theorem, the integral of f along γ is completely determined by the residues of the points within γ.

- If only one point z_1 encircled by γ has a non-zero residue, then

$$\int_\gamma f(z)\,dz = 2\pi i \cdot \mathrm{Res}_{z_1}(f)$$

[This is the definition of the residue]

- If several points with non-zero residue are encircled by γ, say, $z_1, z_2, \ldots, z_m$, then the residues add up

$$\int_\gamma f(z)\,dz = 2\pi i \cdot \sum_{i=1}^{m} \mathrm{Res}_{z_i}(f)$$

- If the points encircled by γ all have zero residue, then

$$\int_\gamma f(z)\,dz = 0,$$

Notice then that f is holomorphic over int γ and so this is just the Cauchy theorem.

Example 1 Consider $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $f(z) = 1/z$

and let γ be a simple closed differentiable curve.

a) If $0 \notin \overline{\operatorname{int}\gamma}$, then $\int_\gamma f(z)\,dz = 0$ [Cauchy theorem / residue theorem]

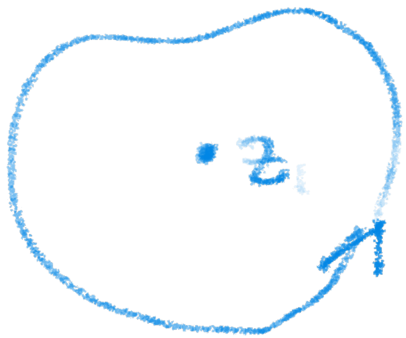
b) If $0 \in \operatorname{int}\gamma$, then

$$\int_\gamma f(z)\,dz = 2\pi i \cdot \operatorname{Res}_0(f) = 2\pi i \cdot 1 = 2\pi i$$

c) If 0 lies on the curve, then the integral is not defined.

a)

b)



Example 2: Let $f: \mathbb{C} \setminus \{0,-1\} \to \mathbb{C}$, $\quad f(z) = \frac{1}{z(z+1)}$

At 0 and -1, we have poles of order 1. We compute the residues:

$$\operatorname{Res}_0(f) = \lim_{z \to 0} z f(z) = \lim_{z \to 0} \frac{1}{z+1} = 1$$

$$\operatorname{Res}_{-1}(f) = \lim_{z \to -1} (z+1) f(z) = \lim_{z \to -1} \frac{1}{z} = -1$$

Having computed the residues, let γ be a closed siple differentiable curve.

Case a) $\quad 0, -1 \notin \overline{\operatorname{int} \gamma} \quad \Rightarrow \quad \int_\gamma f(z)\,dz = 0$

Case b) $\quad 0 \in \operatorname{int} \gamma$
$\quad -1 \notin \overline{\operatorname{int} \gamma} \quad \Rightarrow \quad \int_\gamma f(z)\,dz = 2\pi i \cdot \operatorname{Res}_0(f) = 2\pi i$

Case c) $0 \notin \overline{\operatorname{int}\gamma}$
$-1 \in \operatorname{int}\gamma$

$$\Rightarrow \int_\gamma f(z)\,dz = 2\pi i \operatorname{Res}_{-1}(f) = -2\pi i$$

Case d) $0, -1 \in \operatorname{int}\gamma$

$$\Rightarrow \int_\gamma f(z)\,dz = 2\pi i \left(\operatorname{Res}_0(f) + \operatorname{Res}_{-1}(f) \right)$$

$$= 2\pi i \left(1 + (-1) \right) = 0$$

Case e) if at least one of the singularities lies on the curve, then the integral is not defined.