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I.3. Limits

I.4. Complex derivatives

I.5. Complex logarithm

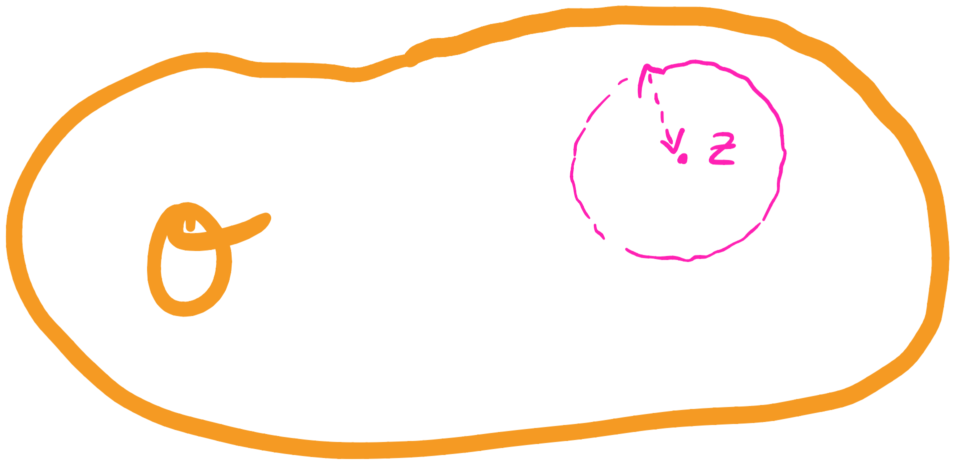
Brief review: open sets

What is an open set?

We call $\mathcal{O} \subseteq \mathbb{C}$ open if $\forall z \in \mathbb{C}\ \exists\, \varepsilon > 0$ such that

$$\{ w \in \mathbb{C} \mid |w - z| < \varepsilon \} \subseteq \mathcal{O}$$

In other words, we can always find, for any $z \in \mathbb{C}$, a small ball around z that lies completely in $\mathcal{O}$



Example: $\mathbb{C}$ is open

I. 3 Limits

Let $\mathcal{O} \subseteq \mathbb{C}$ be open.

A function $f : \mathcal{O} \to \mathbb{C}$ is continuous at $z_0 \in \mathbb{C}$ if

$$\forall \varepsilon > 0 \quad \exists \delta > 0 : \quad |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$$

In that case, we write

$$f(z_0) = \lim_{z \to z_0} f(z)$$

We call $f : \mathcal{O} \to \mathbb{C}$ continuous if it is continuous at all $z_0 \in \mathcal{O}$.

One easily sees that $f = u + iv$ is continuous (at z_0) if and only if $u = \mathrm{Re}\, f$ and $v = \mathrm{Im}\, f$ are continuous (at z_0)

Note: if $f, g : \mathcal{O} \to \mathbb{C}$ are continuous, then so are

$$f + g, \qquad f \cdot g$$

I.4. Complex derivatives

We call $f : \mathcal{O} \to \mathbb{C}$ complex differentiable / holomorphic at $z_0 \in \mathbb{C}$

if $$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$ exists and is finite

We call f holomorphic / complex differentiable if it is differentiable at every $z_0 \in \mathcal{O}$.

[If $f : \mathcal{O} \to \mathbb{C}$ is holomorphic and $\mathcal{O} = \mathbb{C}$, then f is also called entire]

The definition seems analogous to differentiability of functions $f : \mathbb{R} \to \mathbb{R}$, but in the complex setting differentiability is much stronger.

We call $f : \mathbb{C} \to \mathbb{C}$ complex differentiable at $z_0 \in \mathbb{C}$ if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists and is finite

This definition is completely analogous to the definition of derivatives in Analysis I & II.

However, such functions are much stronger, satisfying the

$\Rightarrow$ Cauchy-Riemann Equations

Cauchy-Riemann Equations

Let $\mathcal{O} \subseteq \mathbb{C}$ be an open set and let

$$f : \mathcal{O} \to \mathbb{C}, \qquad z = x + iy \longmapsto u(x,y) + v(x,y) \cdot i$$

be holomorphic over $\mathcal{O}$. Then

$$u, v \in C^1(\mathcal{O}), \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = - \frac{\partial v}{\partial x} \qquad \Big\} \circledast$$

Conversely, if $\circledast$ holds, then f is holomorphic over $\mathcal{O}$.

Whenever $f : \mathcal{O} \to \mathbb{C}$ is holomorphic, we have at each $z_0 \in \mathcal{O}$:

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \frac{\partial u}{\partial x}(x,y) + \frac{\partial v}{\partial x}(x,y) \cdot i$$

$$= \frac{\partial v}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y) \cdot i$$

This already determines the real and imaginary part of f

Brief rationale of Cauchy-Riemann equations. Let $z_0 = x_0 + iy_0$

$$f'(z_0) = \lim_{x \to x_0} \frac{f(x+iy_0) - f(x_0+iy_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} + \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0} i$$

$$= \partial_x u(x_0, y_0) + \partial_x v(x_0, y_0) i$$

$$f'(z_0) = \lim_{y \to y_0} \frac{f(x_0+iy) - f(x_0+iy_0)}{iy - iy_0}$$

$$= \lim_{x \to x_0} \frac{u(x_0, y) - u(x_0, y_0)}{iy - iy_0} + \frac{v(x_0, y) - v(x_0, y_0)}{iy - iy_0} i$$

$$= \partial_y v(x_0, y_0) - \partial_y u(x_0, y_0) i$$

Ex 1

$$f(z) = z^2 = \underbrace{x^2 - y^2}_{u(x,y)} + \underbrace{2xy}_{v(x,y)} \cdot i$$

We check the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Hence $f(z) = z^2$ is holomorphic, and

$$f'(z) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} i = 2x + 2y \cdot i = 2z$$

Ex 2 $\quad f(z) = \bar{z} = x - iy$

$$\frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 0 \qquad \frac{\partial v}{\partial y} = -1$$

$$\boxed{\partial_x u \neq \partial_y v}$$

The Cauchy-Riemann Equations are not satisfied, hence the complex conjugate is not complex differentiable.

Ex 3 : $f : \mathbb{C}\setminus\{0\} \to \mathbb{C}, \quad z \longmapsto 1/z$

We have seen

$$f(z) = \frac{x}{x^2+y^2} + \frac{-y}{x^2+y^2}\, i \qquad x+iy \neq 0$$

We verify C.R.-Eq.

$$\frac{\partial u}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial x} = \frac{2xy}{(x^2+y^2)^2} = -\frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y} = \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial x}$$

Therefore

$$f'(z) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} i = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} + \frac{2xy}{(x^2+y^2)^2} i$$

We suspect that this should equal $-1/z^2$.

$$\frac{1}{z^2} = \frac{1}{(x + yi)^2} = \frac{1}{x^2 - y^2 + 2xy \cdot i}$$

$$= \frac{x^2 - y^2 - 2xy \cdot i}{(x^2 - y^2)^2 + 4x^2y^2} = \frac{x^2 - y^2 - 2xyi}{x^4 + y^4 + 2x^2y^2}$$

$$f'(z) = \frac{-1}{z^2} = \frac{x^2 - y^2 - 2xy \cdot i}{(x^2 + y^2)^2}$$

Ex 4 $\quad f(z) = e^z = e^x \cos(y) + e^x \sin(y)\, i$

$$\frac{\partial u}{\partial x} = e^x \cos(y) \qquad \frac{\partial u}{\partial y} = -e^x \sin(y) = -\frac{\partial v}{\partial x}$$

$$\frac{\partial v}{\partial x} = e^x \sin(y) \qquad \frac{\partial v}{\partial y} = e^x \cos(y) = \frac{\partial u}{\partial x}$$

Cauchy-Riemann equations satisfied, hence f holomorphic, and

$$f'(z) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} i = e^x \cos(y) + e^x \sin(y) = e^z$$

Remark (complex conjugate)

We can directly check that $f(z) = \bar{z}$ is not holomorphic

Suppose that $z_0 = x_0 + y_0 i$ and f is diff'able at z_0

i.e., $f'(z_0) = \lim\limits_{z \to z_0} \dfrac{\bar{z} - \bar{z_0}}{z - z_0}$ exists and is finite

a) Let $z = x + iy_0$ approximates z_0

$$f'(z_0) = \lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1$$

b) Let $z = x_0 + iy$ approximate z_0

$$f'(z_0) = \lim_{y \to y_0} \frac{-yi + y_0 i}{yi - y_0 i} = -1$$

Rules of differentiation for complex derivatives

If $\mathcal{O} \subseteq \mathbb{C}$ is open and $f, g : \mathcal{O} \to \mathbb{C}$ holomorphic, then

$$(f+g)' = f' + g'$$

$$(f \cdot g)' = f' \cdot g + f \cdot g'$$

$$(f/g)' = \frac{f'g - fg'}{g^2} \quad \text{where } g \neq 0$$

Lastly, if $g(\mathcal{O}) \subseteq \mathcal{O}$, then we can compose f and g,

$$(f \circ g)'(z) = f'(g(z)) \cdot g'(z)$$

Made with Goodnotes

Remark We can check this the same manner as for real derivatives. For example

$$\lim_{z\to z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z \longrightarrow z_0}$$

$$= \lim_{z\to z_0} \frac{f(z)g(z) - f(z_0)g(z) + f(z_0)g(z) - f(z_0)g(z_0)}{z - z_0}$$

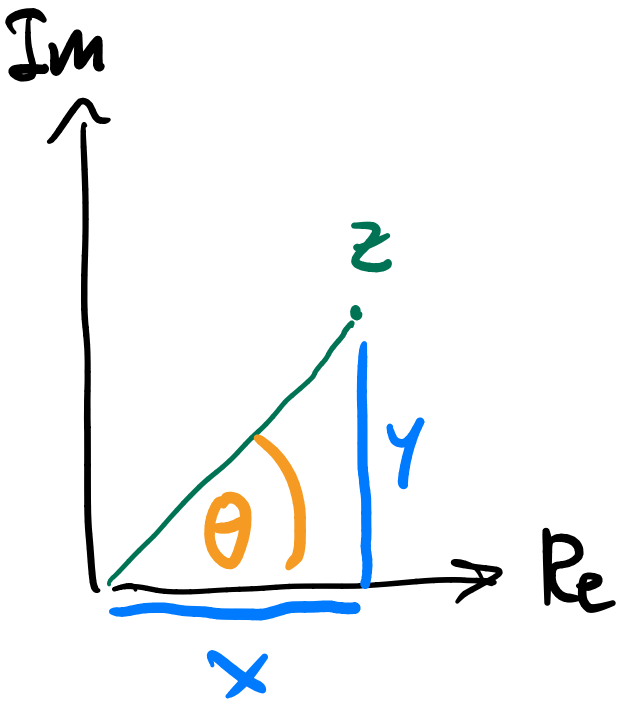
$$= \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} g(z_0) + \lim_{z\to z_0} f(z_0)\frac{g(z) - g(z_0)}{z - z_0}$$

$$= f'(z_0)\cdot g(z_0) + f(z_0)\, g'(z_0)$$

I. 5 Complex logarithm

We recall the principal argument of a complex number

$$-\pi < \theta \leq \pi$$



How to compute the principal argument?

$$\theta = \begin{cases} \arctan(y/x) & x > 0 \\ \pi/2 & x = 0,\ y > 0 \\ \pi + \arctan(y/x) & x < 0,\ y \geq 0 \\ -\pi/2 & x = 0,\ y < 0 \\ -\pi + \arctan(y/x) & x < 0,\ y < 0 \end{cases}$$

To motivate the complex logarithm, suppose

$$z = e^{x+iy} = e^x \cdot e^{iy} = |z| \cdot e^{i \arg(z)}$$

Given z, the complex logarithm should provide $x+iy$

In addition, we want y to be the principle argument

$$-\pi < y \leq \pi$$

Hence, we define

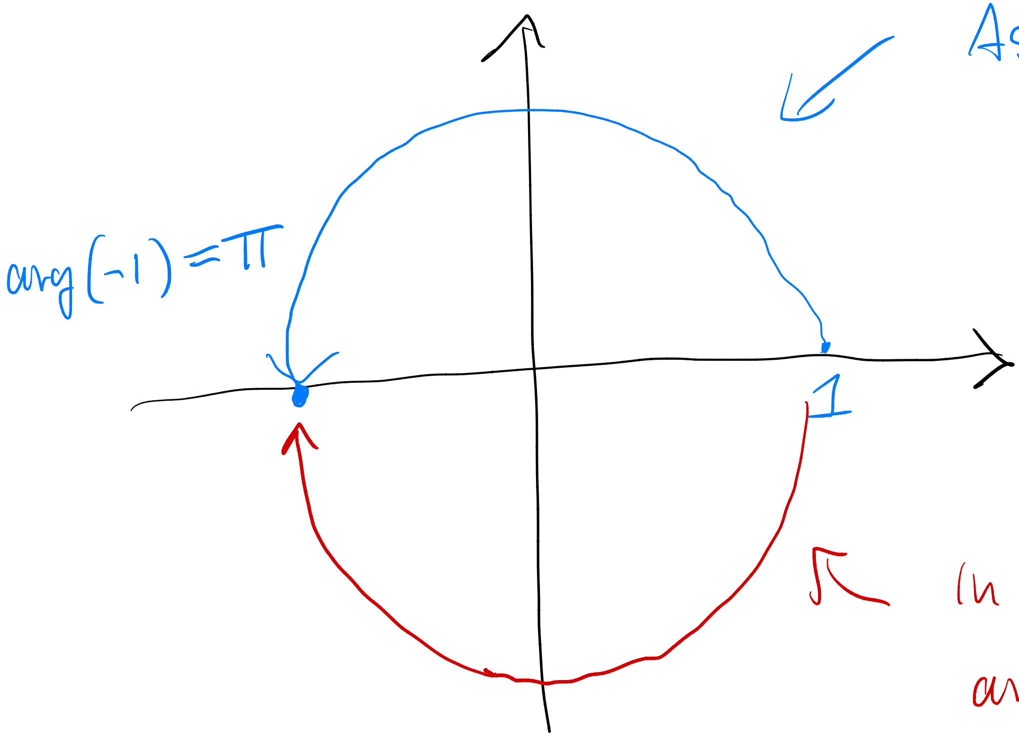
$$\log z = x + iy$$

$$= \underbrace{\log|z|}_{\log(e^x)} + \arg z \cdot i$$

From the definition, it already follows that the complex logorithm is not defined at zero, $z = 0$.

The complex logorithm is not continuous

because $\arg(z)$ is not continuous



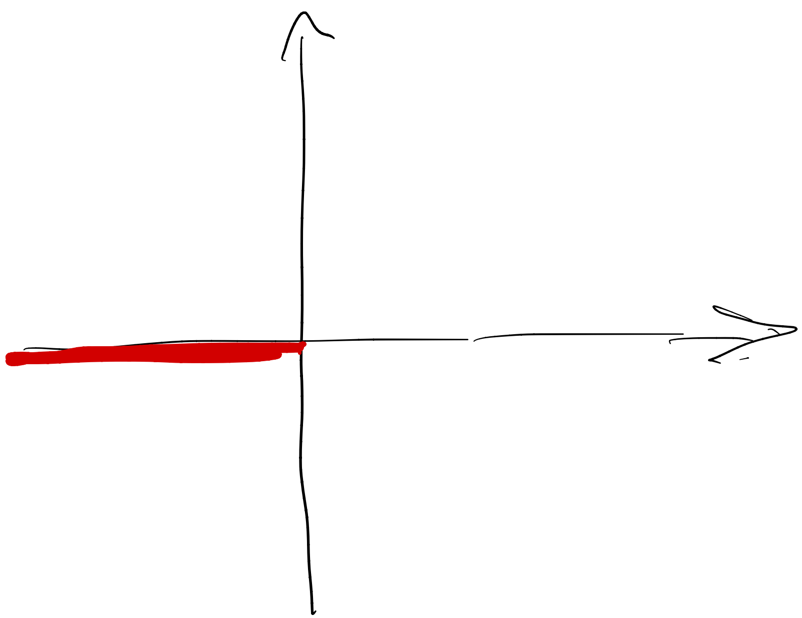
As we walk along the complex unit circle counterclockwise $\arg(z)$ **becomes** π

In the opposite direction, $\arg(z)$ will go down to $-\pi$

Hence, along the negative numbers, the complex logorithm is not continuous. It jumps along the negative numbers.

However, $\log(z)$ is continuous over

$$\mathbb{C} \setminus \{ z \in \mathbb{C} \mid \operatorname{Im}(z) = 0, \ \operatorname{Re}(z) \leq 0 \}$$



What are the real and imaginary components of $f(z) = \log(z)$?

Let us write $f(z) = u(x,y) + v(x,y) \cdot i$. For simplicity, consider $x > 0$. Then

$$\log(z) = \log|z| + \arg(z) \cdot i$$

Writing $z = x + iy$, we find

$$\log(z) = \underbrace{\log\sqrt{x^2+y^2}}_{u(x,y)} + \underbrace{\arctan(y/x)}_{v(x,y)} \cdot i$$

NB: For positive real numbers $z_1, z_2 \in \mathbb{R}$

$$\log(z_1 z_2) = \log(z_1) + \log(z_2) \qquad (*)$$

But for general $z_1, z_2 \in \mathbb{C}$, this no longer holds. For example,

$$z_1 = z_2 = -1 \quad \Rightarrow \quad \arg z_1 = \arg z_2 = \pi$$

We have

$$\log(z_1 z_2) = \log(1) = 0 \quad \neq \quad \log(z_1) + \log(z_2) = 2\pi \cdot i$$

Hence $(*)$ no longer holds for the complex logarithm

With the logarithm, we can define the power function with complex exponent.

Let $\gamma \in \mathbb{C}$.

$$f(z) := z^{\gamma} := e^{\log(z^{\gamma})} := e^{\gamma \cdot \log(z)}$$

$$= e^{\gamma \log|z| + \gamma \arg(z) \cdot i}$$

The power function with exponent γ is holomorphic over

$$\mathbb{C} \setminus \{ z \in \mathbb{C} \mid \operatorname{Im}(z) = 0, \operatorname{Re}(z) \leq 0 \}$$