20.02.25

I Complex-valued functions, Cauchy-Riemann Equations

I.1 Review complex number

I.2 Complex functions

I. Complex-valued fonctions and Cauchy-Riemann equations

I.1. Review of complex numbers $\mathbb{C}$

- Imaginary unit i : $i^2 = -1$
- All complex numbers have the form

$$z = x + iy \in \mathbb{C}, \quad x, y \in \mathbb{R}$$

$x = \operatorname{Re} z$ $\quad\quad$ $y = \operatorname{Im} z$

real part $\quad\quad$ imaginary part

- $\bar{z} = x - iy$ (complex conjugate)

  change the sign of the imaginary part

- $|z| = \sqrt{x^2 + y^2}$ (absolute value / modulus)

- $e^{i\theta} := \cos(\theta) + \sin(\theta) \cdot i, \quad \theta \in \mathbb{R}$

We compute with complex numbers as follows:

$$z_1 = x_1 + iy_1, \qquad z_2 = x_2 + iy_2$$

- $z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$
- $z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2)i$
- $z_1 \cdot z_2 = (x_1 + y_1 i) \cdot (x_2 + y_2 i)$

$$= x_1 x_2 + x_1 y_2 i + y_1 x_2 i + y_1 y_2 i^2$$

$$= \underbrace{(x_1 x_2 - y_1 y_2)}_{\text{Real part}} + \underbrace{(x_1 y_2 + y_1 x_2)}_{\text{Imaginary part}} i$$

- $z \cdot \overline{z} = (x+iy)(x-iy) = x^2 + y^2 = |z|^2$

- $z_1/z_2 = \dfrac{z_1 \cdot \overline{z_2}}{z_2 \cdot \overline{z_2}} = \dfrac{z_1 \cdot \overline{z_2}}{|z_2|^2}$

  provided that $z_2 \neq 0$

Now revisit the complex exponential:

If $z \in \mathbb{C}$, then we con write

$$z = |z| \cdot e^{i\theta}$$

for some $\theta \in \mathbb{R}$, which we call the <u>argument</u>.

How is that representation of complex numbers useful?

<u>Example:</u> Since $z = |z| \cdot e^{i\theta}$,

we have $z^k = |z|^k e^{ik\theta}$

<u>Example:</u> Suppose $z_1 = |z_1| e^{i\theta_1}$, $z_2 = |z_2| e^{i\theta_2}$

for some $\theta_1, \theta_2 \in \mathbb{R}$

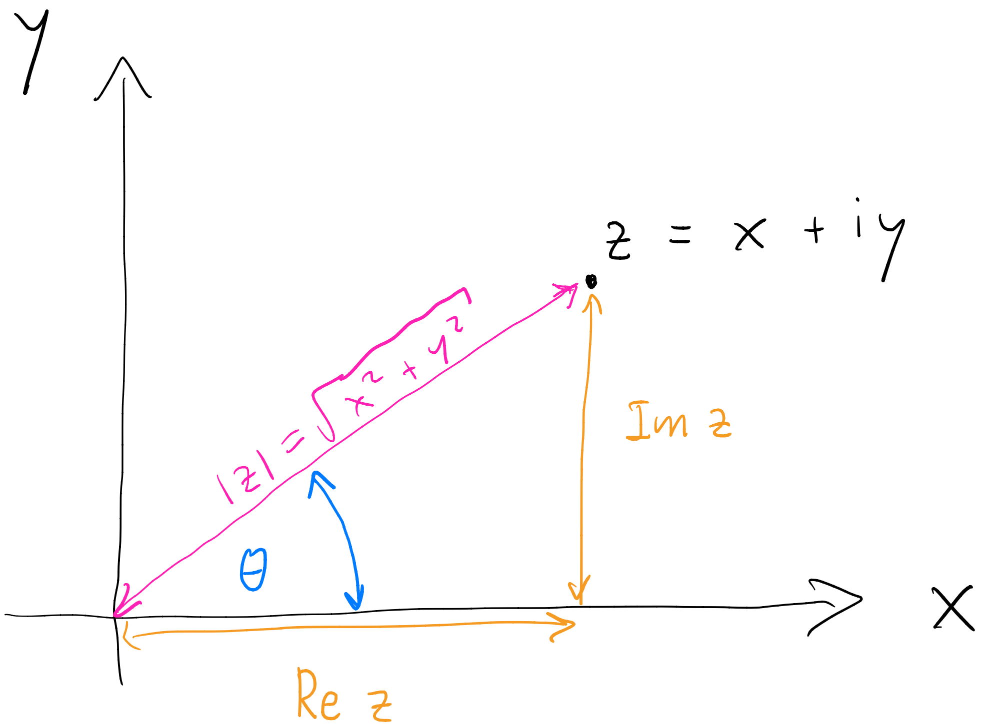
$$z_1 \cdot z_2 = |z_1| e^{i\theta_1} |z_2| e^{i\theta_2} = |z_1| \cdot |z_2| e^{i(\theta_1 + \theta_2)}$$

$$z_1 / z_2 = \frac{|z_1| \cdot e^{i\theta_1}}{|z_2| \cdot e^{i\theta_2}} = \frac{|z_1|}{|z_2|} e^{i(\theta_1 - \theta_2)}$$

provided $z_2 \neq 0$.

These formulas are intuitively clear, and can be verified with our definitions of $e^{i\theta}$, the complex product, and the complex division.

Geometric interpretation of $\mathbb{C}$



We can interpret $z = x + iy \in \mathbb{C}$ as a two component vector.

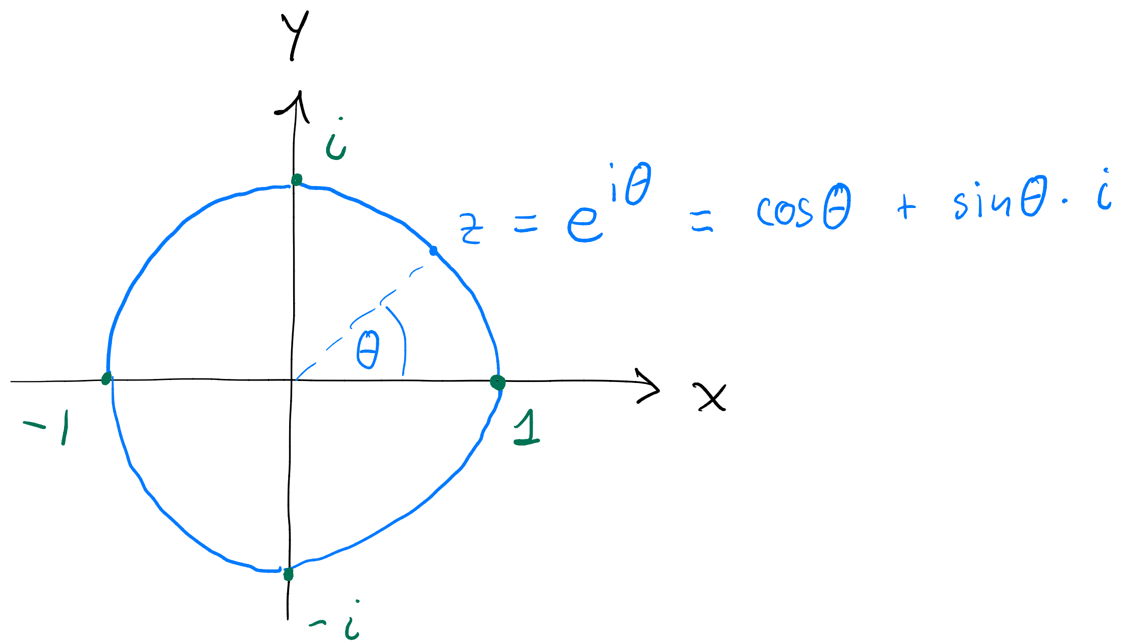
The absolute value $|z|$ is the length of that vector.

The argument θ is the angle of z with the x-axis.

In this context, the x-axis is also called 'real axis' and the y-axis is called 'imaginary axis'.

The complex numbers $z \in \mathbb{C}$ with modulus $|z| = 1$ constitute the so-called <u>complex unit circle</u>. Every $z \in \mathbb{C}$ with $|z| = 1$ can be written

$$z = \cos\theta + \sin\theta \cdot i \quad \text{for some } \theta \in \mathbb{R}$$
$$= e^{i\theta}$$



When writing

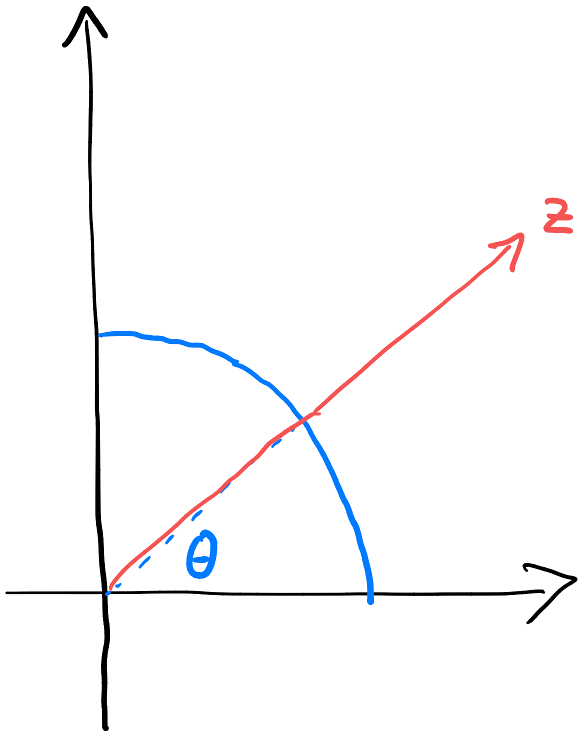
$$z = |z| \cdot e^{i\theta},$$

we may think

$e^{i\theta}$ = direction

$|z|$ = length

Example



$$z = 2e^{i\pi/4} \qquad \theta = \pi/4, \quad |z| = 2$$

$$= 2\left(\cos \pi/4\right) + 2\left(\sin \pi/4 \, i\right)$$

$$e^{0} = 1, \quad e^{\frac{\pi}{2} i} = i$$

$$e^{\pi i} = -1, \quad e^{\frac{3}{2}\pi i} = -i$$

$$e^{2\pi i} = 1$$

Generally speaking, the argument is not unique.

The argument θ in the representation

$$z = |z| \cdot e^{i\theta}$$

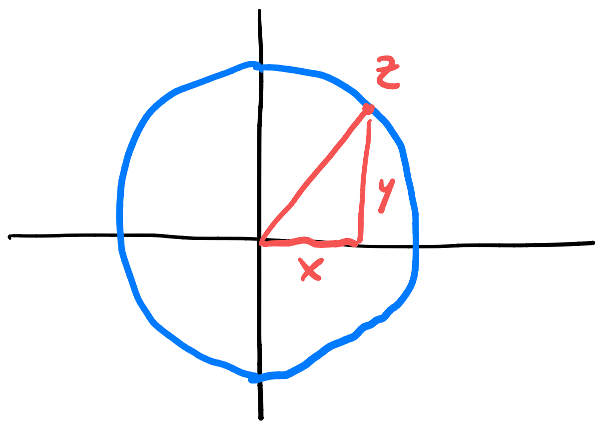
is only unique up to $2\pi \cdot \mathbb{Z}$, that is, integer multiples of 2π

The argument with $-\pi < \theta \leq \pi$ is called the <u>principal argument</u>

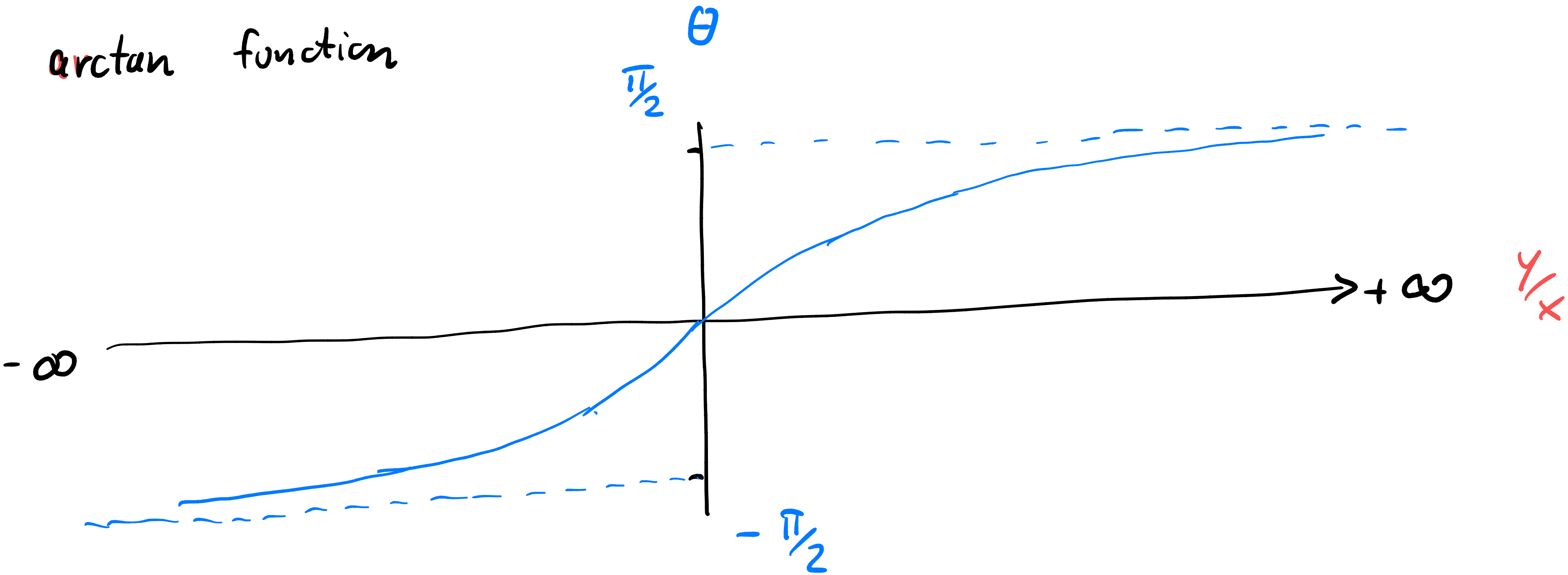
How to find the principal argument?

In the special case $x > 0$, we have $\tan\theta = \frac{y}{x}$, hence

$$\arctan(y/x) = \theta$$



arctan function



II. 2. Complex functions

We study functions $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto f(z)$

$$f(z) = f(x+iy) = u(x,y) + v(x,y) \cdot i$$

Examples

1) $f(z) = z = x + iy$

$u(x,y) = x$
$v(x,y) = y$

2) $f(z) = \bar{z} = x - yi$

$u(x,y) = x$
$v(x,y) = -y$

3) $f(z) = z^2 = (x+iy)^2$

$$= \underbrace{(x^2 - y^2)}_{u(x,y)} + \underbrace{2xy}_{v(x,y)} \cdot i$$

4) $$f(z) = \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$= \underbrace{\frac{x}{x^2+y^2}}_{u(x,y)} + \underbrace{\left(\frac{-y}{x^2+y^2}\right)}_{v(x,y)} i$$

5) $$f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy}$$

$$= \underbrace{e^x \cos(y)}_{u(x,y)} + \underbrace{e^x \sin(y)}_{v(x,y)} i$$

"Official" definition of the exponential function:

$$f(z) = e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$$

$$= 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \dots$$

We have the formula as for exponentials of real numbers. One can show that $e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$ still works

6) $$\cos(z) = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2} \qquad \sinh(z) = \frac{e^{z} - e^{-z}}{2}$$

We can explicitly compute $u = \text{Re } f$ and $v = \text{Im } f$ for each of these functions. For example:

$$f(z) = \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$= \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)} \right)$$

$$= \frac{1}{2}\left(e^{-y} e^{ix} + e^{y} e^{-ix} \right)$$

$$= \frac{1}{2}\left(e^{-y}\left(\cos x + i \sin(x)\right) + e^{y}\left(\cos x - i \sin(x) \right) \right)$$

$$= \frac{1}{2} e^{-y} \cos x + \frac{1}{2} e^{-y} \sin(x) \cdot i + \frac{1}{2} e^{y} \cos(x) - \frac{1}{2} e^{y} \sin(x)\, i$$

$$= \frac{1}{2} \cos x \left(e^{y} + e^{-y} \right) + \frac{1}{2}\left[e^{-y} - e^{y} \right] \sin(x) \cdot i$$

$$= \underbrace{\cos x \cdot \cosh y}_{u(x,y)} + \underbrace{-\sin x \cdot \sinh y}_{v(x,y)} \cdot i$$