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Laplace transform

Example II

We compute Laplace transform

$f : \mathbb{R}_0^+ \longrightarrow \mathbb{R}, \qquad f(t) = t$

By definition

$$F(z) = \int_0^\infty t e^{-zt}\, dt = \left[\frac{t e^{-zt}}{-z} \right]_{t=0}^{t=\infty} - \int_0^\infty \frac{e^{-zt}}{-z}\, dt$$

integration by parts

$$= \left[\frac{t e^{-zt}}{-z} \right]_{t=0}^{t=\infty} - \left[\frac{e^{-zt}}{z^2} \right]_{t=0}^{t=\infty}$$

Here, $t = \infty$ is to be interpreted as a limit.

$$F(z) = \lim_{t\to\infty} \frac{te^{-zt}}{-z} - \cancel{\frac{0 \cdot e^{-z\cdot 0}}{-z}} - \lim_{t\to\infty} \frac{e^{-zt}}{z^2} + \frac{e^{-z\cdot 0}}{z^2}$$

If $\operatorname{Re} z > 0$, then the limits go to zero, and so

$$F(z) = 1/z^2$$

Domain of convergence : $\{ z \in \mathbb{C} \mid \operatorname{Re} z > 0 \}$

Example III We search the Laplace transform of

$$f : \mathbb{R}_0^+ \to \mathbb{C}, \qquad f(t) = e^{at} \qquad a \in \mathbb{C}$$

By definition

$$F(z) = \int_0^\infty e^{-zt} f(t)\, dt = \int_0^\infty e^{-zt} e^{at}\, dt$$

$$= \int_0^\infty e^{(a-z)t}\, dt = \left[\frac{e^{(a-z)t}}{a-z} \right]_{t=0}^{t=\infty} \qquad \text{if } a \neq z$$

$$= \lim_{t\to\infty} \frac{e^{(a-z)t}}{a-z} - \frac{e^{(a-z)\cdot 0}}{a-z}$$

Whether the limit converges depends on a. If $\mathrm{Re}(z) > \mathrm{Re}(a)$, then

$$\lim_{t\to\infty} \frac{e^{(a-z)t}}{a-z} = 0 \qquad \text{exponential decay}$$

In that case,

$$F(z) = -\frac{e^{(a-z)\cdot 0}}{a-z} = \frac{1}{z-a}$$

We notice that the formula for $F(z)$ is often defined not only over the domain of convergence. Nevertheless, we typically use the Laplace transform over the domain of convergence.

Depending on the textbook author, "domain of convergence" may refer to any domain where the Laplace transform is well-defined, or the largest such domain.

Definition: Any real number $\gamma \in \mathbb{R}$ for which

$$\{ z \in \mathbb{C} \mid \operatorname{Re} z > \gamma \}$$

lies within the domain of convergence of the Laplace transform $F(z)$ is called an "abcyssa of convergence".

That is, if $\operatorname{Re}(z) > \gamma$ is above that threshold, then convergence of $F(z)$ is guaranteed.

Inverse Laplace transform:

The Laplace transform is one-to-one: $\mathcal{L}(f)$ determines f, that is, the transform of the signal already determines the original signal.

In other words: $\mathcal{L}(f) = \mathcal{L}(g) \implies f = g$

We recover the original signal from its Laplace transform via the inverse Laplace transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\gamma + i \cdot s)\, e^{(\gamma + is)t}\, ds$$

where γ is large enough, namely an abcyssa of convergence.

The integral will be independent of γ, as long as it is large enough.

The formula holds for all $t \geq 0$, which is the range of t for which the original signal is defined.

This formula can be derived via the Fourier transform
(no details this semester)

Practically, we will rarely use this formula, but instead use the Laplace transform table.

Properties of the Laplace transform

- Linearity $\mathcal{L}(a \cdot f + b \cdot g) = a \cdot \mathcal{L}(f) + b \cdot \mathcal{L}(g)$,

  for $a, b \in \mathbb{C}$.

Phase change If $a, b \in \mathbb{R}$ with $a > 0$, then

$$g(t) = e^{-bt} f(at) \implies G(z) = \frac{1}{a} F\left(\frac{z+b}{a}\right)$$

That describes how to modify the Laplace transform if
- we rescale the variable t by $a > 0$
- we remodulate the signal by an exponential factor.

Interaction with derivatives

$$\mathcal{L}(f') = z\,\mathcal{L}(f) - f(0)$$
$$= z\,F(z) - f(0)$$

$\boxed{\text{Can be seen via integration by parts}}$

More generally, we can repeat this:

$$\mathcal{L}(f'') = z\,\mathcal{L}(f') - f'(0)$$
$$= z\left[z\,\mathcal{L}(f) - f(0)\right] - f'(0)$$
$$= z^2\,\mathcal{L}(f) - z\,f(0) - f'(0)$$

Similarly,

$$\mathcal{L}(f''') = z^3\,\mathcal{L}(f) - z^2 f(0) - z\,f'(0) - f''(0)$$

We can iterate this formula:

$$\mathcal{L}(f^{(n)}) = z^n \mathcal{L}(f) - z^{n-1} f(0) - z^{n-2} f'(0) - \dots$$
$$- z f^{(n-2)}(0) - f^{(n-1)}(0)$$

The Laplace transform converts derivatives into polynomials.

<u>Integration</u>:

$$g(t) = \int_0^t f(s)\,ds \implies G(z) = \frac{F(z)}{z}$$

These formulas can be found in textbooks, we will not prove them.

Application to the Poisson problem

We want to solve the boundary value problem:

$$- u''(x) + k^2 u(x) = 0, \qquad a < x < b$$

$$u(a) = g_a, \qquad u(b) = g_b,$$

for some boundary values $g_a, g_b \in \mathbb{R}$, and $k > 0$.

For simplicity, consider the case $a = 0$.

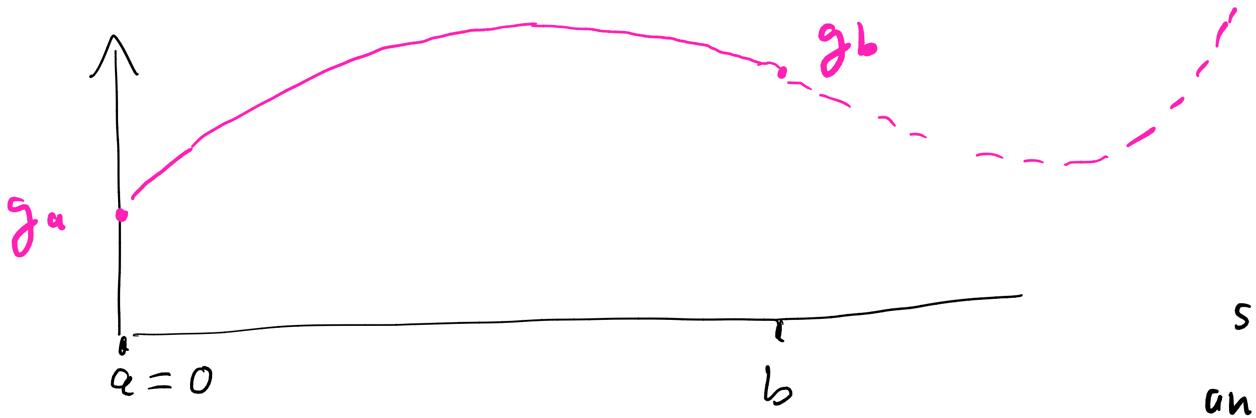
To solve the problem with the Laplace transform, we search for

$$u : [0, \infty) \longrightarrow \mathbb{R}$$

with

$$- u''(x) + k^2 u(x) = 0, \qquad 0 < x$$

$$u(0) = g_a, \qquad u(b) = g_b$$



We approach the boundary value problem as an initial value problem starting at $a=0$ and having an intermediate value fixed at b.

We apply the Laplace transform to the differential equation

$$-u''(x) + k^2 u(x) = 0$$

$$-\mathcal{L}[u''](z) + k^2 \mathcal{L}[u](z) = 0$$

for any z in the domain of convergence.

We recall: $\mathcal{L}[u''] = z^2 \mathcal{L}[u] - z\,u(0) - u'(0)$

Hence:

$$- z^2 \mathcal{L}[u](z) + z u(0) + u'(0) + k^2 \mathcal{L}[u](z) = 0$$

We isolate the Laplace transform:

$$(k^2 - z^2) \mathcal{L}[u](z) = - z u(0) - u'(0)$$

$$\mathcal{L}[u](z) = \frac{z u(0) + u'(0)}{z^2 - k^2}$$

[provided that z is in the domain of convergence]

$$\mathcal{L}[u](z) = u(0) \frac{z}{z^2 - k^2} + u'(0) \frac{1}{z^2 - k^2}$$

We have a look at the Laplace transform table:

$$\sinh(\omega t) \rightsquigarrow \frac{\omega}{z^2-\omega^2}, \qquad \cosh(\omega t) \rightsquigarrow \frac{z}{z^2-\omega^2}$$

$$\sinh(x) = \tfrac{1}{2}(e^x - e^{-x}), \qquad \cosh(x) = \tfrac{1}{2}(e^x + e^{-x})$$

In our situation, we recognize

$$\frac{u'(0)}{k}\sinh(kt) \rightsquigarrow \frac{u'(0)}{z^2-k^2}$$

$$u(0)\cosh(kt) \rightsquigarrow u(0)\frac{z}{z^2-k^2}$$

These are the inverse Laplace transforms of the two terms in the definition of $\mathcal{L}[u]$

This leads to

$$u(t) = \frac{u'(0)}{k} \sinh(kt) + u(0) \cosh(kt)$$

We need to find $u(0)$ and $u'(0)$ as initial values to match the desired values of u at $t=a$ and $t=b$.

We can already pick $u(0) = g_a$ ✓

We want to fix $u'(0)$ so that u will hit the desired value g_b at $t=b$.

We use the formula:

$$u(t) = \frac{u'(0)}{k} \sinh(kt) + g_a \cosh(kt)$$

$$u(b) = \frac{u'(0)}{k} \sinh(kb) + g_a \cosh(kb)$$

We know that $\sinh(kb) \neq 0$, and hence we can isolate $u'(0)$ in the formula above.

$$u(b) - g_a \cosh(kb) = u'(0) \frac{\sinh(kb)}{k}$$

$$\frac{k}{\sinh(kb)} \Big(\underset{g_b}{u(b)} - g_a \cosh(kb) \Big) = u'(0)$$

In summary we have fixed $u(0) = g_a$ immediately, and then we have fixed $u'(0)$ to ensure $u(b) = g_b$.

That produces the complete formula for the solution of the boundary value problem, using only g_a, g_b, and k.

Lastly, we rename t to x, and we are done.

VII Application of the Laplace transform to the Cauchy problem.