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Applications of residue theorem

Fourier transform, recap

Laplace transform

<u>Example 4</u> Suppose we compute the integral

$$I := \int_0^{2\pi} \frac{P(\cos t, \sin t)}{Q(\cos t, \sin t)}\, dt$$

where $P(x,y)$ and $Q(x,y)$ such that $Q(\cos t, \sin t) \neq 0$ for any $0 \leq t \leq 2\pi$.

Recall that

$$\cos t = \frac{e^{it} + e^{-it}}{2}, \qquad \sin t = \frac{e^{it} - e^{-it}}{2i}, \qquad 1 = \frac{1}{ie^{it}} \cdot ie^{it}$$

Now

$$I := \int_0^{2\pi} \frac{P\left(\frac{1}{2}\left(e^{it} + \frac{1}{e^{it}}\right), \frac{1}{2i}\left(e^{it} - \frac{1}{e^{it}}\right)\right)}{Q\left(\frac{1}{2}\left(e^{it} + \frac{1}{e^{it}}\right), \frac{1}{2i}\left(e^{it} - \frac{1}{e^{it}}\right)\right)} \frac{1}{ie^{it}} \cdot ie^{it}\, dt$$

Using the parameterization $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = e^{it}$,

we recognize the curve integral

$$I = \int_\gamma \underbrace{\frac{P\left(\frac{1}{2}(z + \frac{1}{z}), \frac{1}{2i}(z - \frac{1}{z})\right)}{Q\left(\frac{1}{2}(z + \frac{1}{z}), \frac{1}{2i}(z - \frac{1}{z})\right)} \frac{1}{iz}}_{f(z)} \, dz$$

$$= \int_\gamma f(z)\, dz$$

If f is holomorphic except for singularities over the open unit disk, then we apply the residue theorem to get I.

Hence, $z_1, z_2, \dots, z_m$ are the singular points of f within the unit disk, then

$$I = \int_0^{2\pi} \frac{P(\cos t, \sin t)}{Q(\cos t, \sin t)} = \int_\gamma f(z)\, dz$$

$$= 2\pi i \left(\operatorname{Res}_{z_1}(f) + \operatorname{Res}_{z_2}(f) + \dots + \operatorname{Res}_{z_m}(f) \right)$$

<u>Example 5</u> We compute the integral

$$I := \int_0^{2\pi} \frac{1}{2 + \cos\theta}\, d\theta$$

$P(x,y) = 1$

$Q(x,y) = 2 + x$

Applying the general idea of the preceding example:

$$I = \int_\gamma f(z)\,dz, \qquad f(z) = \frac{1}{iz} \cdot \frac{1}{2 + \frac{1}{2}(z + 1/z)}$$

We find

$$f(z) = \left(2iz + \frac{1}{2} i z^2 + \frac{1}{2} i \right)^{-1}$$

$$= \frac{2}{i(z^2 + 4z + 1)}$$

The singular points of f are the roots of $z^2 + 4z + 1$:

$$z_1 = -2 + \sqrt{3}, \qquad z_2 = -2 - \sqrt{3}$$

$$z^2 + 4z + 1 = (z - z_1)(z - z_2)$$

We only use the singular point(s) inside the unit disk

$$z_1 \approx -0.2679\ldots \checkmark \qquad z_2 \approx -3.732\ldots \times$$

Hence

$$\int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta = \int_\gamma f(z)\, dz = 2\pi i \cdot \operatorname{Res}_{z_1}(f)$$

we see that z_1 is a pole of order 1, therefore

$$\operatorname{Res}_{z_1}(f) = \lim_{z \to z_1} (z - z_1) f(z) = \lim_{z \to z_1} \frac{2}{i}(z - z_2)^{-1}$$

$$= \frac{2}{i}\left(z_1 + 2 + \sqrt{3}\right)^{-1}$$

$$= \frac{2}{i}\left(\cancel{-2} + \sqrt{3} + \cancel{2} + \sqrt{3}\right)^{-1} = \frac{1}{\sqrt{3} \cdot i}$$

Thus,

$$\int_0^{2\pi} (2 + \cos\theta)^{-1} d\theta = \frac{2\pi i}{\sqrt{3} \cdot i} = \frac{2}{\sqrt{3}}\pi$$

In summary:

- Residue theorem to compute integrals along closed curves
- At the end, we compute residues
- Even though a concept of complex analysis, many integrals in real analysis can be computed with a "detour" through complex analysis
- More applications: Fourier analysis

V: Fourier transform, recap

Suppose $f: \mathbb{R} \to \mathbb{C}$ is a function sufficiently regular (e.g. integrable)

The Fourier transform (FT) of f is defined via

$$\hat{f}(\alpha) = \mathcal{F}[f](\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-i\alpha x}\, dx$$

Similarly, we define the inverse Fourier transform

$$\check{f}(x) = \mathcal{F}^{-1}[f](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(\alpha)\, e^{i\alpha x}\, d\alpha$$

As the name already indicates, the inverse FT of a FT of a function f is again the original function f.

FT switches the function from time representation to frequency representation.

time-domain t

space-domain x

$$\begin{tikzcd} f \arrow[r, bend left, "FT"] & \hat{f} \arrow[l, bend left, "\text{inv } FT"] \end{tikzcd}$$

frequency domain

1) The Fourier transform is linear

$$\mathcal{F}[a \cdot f + b \cdot g] = a \hat{f} + b \cdot \hat{g}. \qquad a, b \in \mathbb{C}$$

2) (Time shift) Let $g(x) = f(x - a)$. Then

$$\hat{g}(\alpha) = e^{-i\alpha \cdot a} \cdot \hat{f}(\alpha)$$

3) (Frequency shift) Let $g(x) = e^{ibx} f(x)$

$$\hat{g}(\alpha) = \hat{f}(\alpha - b)$$

4) (Dilation) Let $g(x) = f(cx)$ and $c \neq 0$. Then

$$\hat{g}(\alpha) = \frac{1}{|c|} \hat{f}\left(\frac{\alpha}{c}\right)$$

5) Interaction of FT with derivatives

Let $g(x) = f'(x)$. Then $\hat{g}(x) = i\alpha \hat{f}(\alpha)$

More generally,

if $g(x) = f^{(n)}(x)$, then $\hat{g}(\alpha) = (i\alpha)^n \hat{f}(\alpha)$

n-th derivative

<u>Convolution</u>: given $f, g : \mathbb{R} \longrightarrow \mathbb{C}$, we define:

$$(f * g)(x) := \int_{-\infty}^{+\infty} f(y)\, g(x-y)\, dy .$$

This is another function from $\mathbb{R}$ to $\mathbb{C}$, called the <u>convolution</u> of f and g.

Important properties:

i) $f * g = g * f$ (commutative)

ii) $f * (g * h) = (f * g) * h$ (associative)

iii) $f * (g + h) = f * g + f * h$ (distributive)

6) Interaction of FT and convolutions

Let $f, g : \mathbb{R} \to \mathbb{C}$, then

$$\tilde{\mathcal{F}}[f * g](\alpha) = \sqrt{2\pi}\left(\tilde{\mathcal{F}}[f] \cdot \tilde{\mathcal{F}}[g]\right)(\alpha)$$

That is, the FT transforms convolutions into products.

In applications (differential equations), we often express the FT of some unknown function $u : \mathbb{R} \to \mathbb{C}$ as a product,

$$\hat{u} = \hat{f} \cdot \hat{g}$$

and then we obtain $u : \mathbb{R} \to \mathbb{C}$ by "inverting" that formula.

7) Theorem of Plancherel

Given $f : \mathbb{R} \to \mathbb{C}$, we have

$$\int_{-\infty}^{+\infty} |f(x)|^2 \, dx = \int_{-\infty}^{+\infty} |\hat{f}(\alpha)|^2 \, d\alpha$$

Physical interpretation: the "energy" of the signal f equals the "energy" of its frequency representation.

Interaction of FT and complex analysis

Let $f(x) = (1 + x^2)^{-1}$. Then

$$\hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-i\alpha x}}{1 + x^2} \, dx$$

We apply the residue theorem to find $\hat{f}(\alpha)$ using a previous example.

Let $\alpha \leq 0$. We know

$$\frac{e^{-i\alpha x}}{1 + x^2} = \frac{e^{-i\alpha x}}{(x - i)(x + i)}$$

This leads to:

use the residue in upper half-plane

$$\int_{-\infty}^{+\infty} \frac{e^{-i\alpha x}}{1 + x^2} \, dx = 2\pi i \cdot \mathrm{Res}_i \left[\frac{e^{-i\alpha z}}{1 + z^2} \right]$$

$$= 2\pi i \lim_{z \to i} (z-i) \frac{e^{-i\alpha z}}{1+z^2}$$

$$= 2\pi i \lim_{z \to i} \frac{e^{-i\alpha z}}{z+i} = 2\pi i \frac{e^{-i\alpha i}}{2i} = \pi e^{\alpha}$$

Hence $\alpha \leq 0$ gives us

$$\tilde{\mathcal{F}}\left[(1+x^2)^{-1} \right](\alpha) = \frac{1}{\sqrt{2\pi}} \pi e^{\alpha} = \sqrt{\frac{\pi}{2}} e^{-|\alpha|}$$

Conversely, for $\alpha \geq 0$ we can use symmetry argument, using a change of variables, and find

$$\tilde{\mathcal{F}}\left[(1+x^2)^{-1} \right](\alpha) = \sqrt{\frac{\pi}{2}} e^{-\alpha} = \sqrt{\frac{\pi}{2}} e^{-|\alpha|}$$

Thus, we recover a formula from the FT table.

VI Laplace transform

<u>Definition</u> Let $f: \mathbb{R}_0^+ \longrightarrow \mathbb{C}$ be a function on the non-negative real numbers.

The Laplace transform $F(z)$ of the function $f(t)$ is

$$\mathcal{L}[f](z) := F(z) = \int_0^\infty e^{-zt} f(t)\, dt$$

Typically, the Laplace transform is only defined for certain $z \in \mathbb{C}$ where $\operatorname{Re} z$ is large enough. The largest domain for $F(z)$ is well-defined is called the domain of convergence

$$\left[F(z) = \int_0^\infty e^{-\operatorname{Re}(z) \cdot t} e^{-\operatorname{Im}(z) \cdot t} f(t)\, dt \right]$$

Example 1 We compute the Laplace transform of

$$f : \mathbb{R}_0^+ \to \mathbb{R}, \qquad f(t) = 1$$

By definition,

$$F(z) = \int_0^\infty f(t)\, e^{-zt}\, dt = \int_0^\infty e^{-zt} = \left[-\frac{e^{-tz}}{z} \right]_{t=0}^{t=\infty}$$

$$= \underbrace{\lim_{t\to\infty} \frac{-e^{-tz}}{z}}_{=0} - \frac{-1}{z} = \frac{1}{z}$$

provided that $\operatorname{Re} z > 0$, to make the last limit converge.

Domain of convergence: $\{ z \in \mathbb{C} \mid \operatorname{Re} z > 0 \}$