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VII. 3 Review of convolutions

We recall convolutions of functions from $\mathbb{R}$ to $\mathbb{R}$:
Given $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$, the function

$$(f * g)(x) := \int_{-\infty}^{+\infty} f(y)\, g(x-y)\, dy$$

is called the convolution of f and g. Equivalently,

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-y)\, g(y)\, dy$$

Some algebraic properties:

1. commutativity: $f * g = g * f$

2. associativity: $(f * g) * h = f * (g * h)$

3. linearity / distributivity: for any $\alpha, \beta \in \mathbb{R}$

$$(\alpha f + \beta g) * h = \alpha (f * h) + \beta (g * h)$$

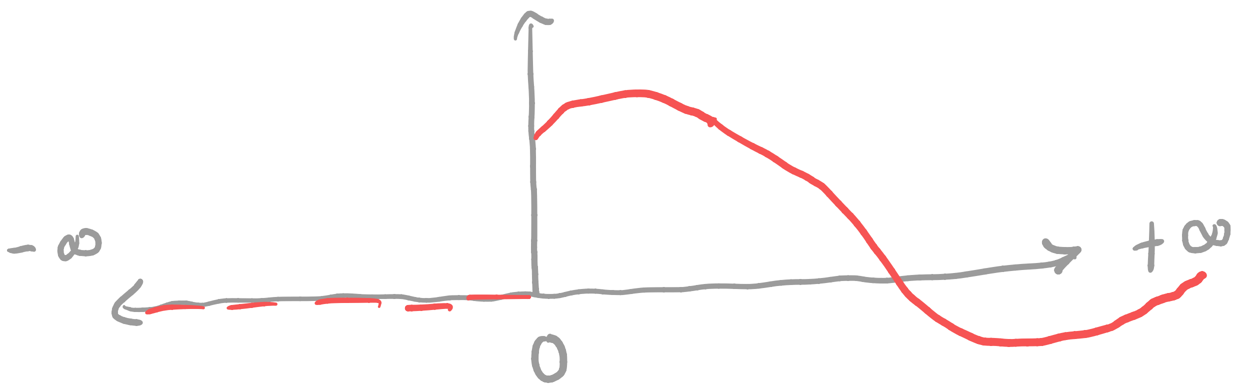
Applications: in many cases, the convolution is a weighted average, e.g.,

$$g := \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 0 \end{cases}$$

Keyword: moving average

The Laplace transform takes functions $f : [0, \infty) \to \mathbb{C}$ and produces functions $F : \mathcal{O} \to \mathbb{C}$ defined over an open subset $\mathcal{O} \subseteq \mathbb{C}$ of the complex plane.

To address convolutions, we extend any such function $f : [0, \infty) \to \mathbb{C}$ to a function $f : (-\infty, +\infty) \to \mathbb{C}$ by zero.



Suppose we have functions $f, g : [0, \infty) \longrightarrow \mathbb{C}$ and extend them by zero to $\mathbb{R}$.

$$(f * g)(t) = \int_{-\infty}^{+\infty} f(s)\, g(t-s)\, ds$$

$f(s) = 0$ for $s < 0$
$$= \int_{0}^{+\infty} f(s)\, g(t-s)\, ds$$

$g(t-s) = 0$ for $t < s$
$$= \int_{0}^{t} f(s)\, g(t-s)\, ds$$

For these particular functions, the convolution is an integral over a finite interval.

Application to Laplace transform:

$$\mathcal{L}[f * g] = \mathcal{L}[f] \cdot \mathcal{L}[g]$$

where $f, g : [0, \infty) \longrightarrow \mathbb{C}$ and $f * g$ is the convolution.

* We have used this for the solution of the Cauchy problem in 1st and 2nd order case
* We will use it for the solution of time-dependent partial differential equations (heat, wave equation)
* In the context of the Laplace transform, the convolution formula simplifies.

VIII Applications to partial differential equations

We study the heat equation as a prototypical time-dependent partial differential equation (PDE).

We search a function with 2nd derivatives solving:

1) $\partial_t u(x,t) = \partial_{xx}^2 u(x,t) + f(x,t), \qquad 0 < x < L, \quad t > 0$

2) $u(x,0) = u_0(x), \qquad 0 < x < L$

3) $u(0,t) = 0, \quad u(L,t) = 0 \qquad t > 0$

Here, $u : [0, L] \times \mathbb{R}_0^+ \to \mathbb{R}$ is the unknown function in position $x \in [0, L]$ and time $t > 0$.

$f(x,t)$ is the source term at position x and time t

$u_0(x)$ describes the initial data at position x at time $t = 0$

For all times $t > 0$, the boundary values at $x = 0$ and $x = L$ are fixed to zero.

The solution theory combines concepts from the Poisson problem and the Cauchy problem (Fourier series, Fourier transform, Laplace transform)

Physical background

The heat equation above describes the diffusion of heat over time starting with an initial distribution u_0 at time $t = 0$.

We assume the temperature fixed at the endpoints (for simplicity).

The source term $f(x,t)$ describes additional sources or sinks of heat, e.g. fire.

The heat flows in the opposite direction of the heat gradient

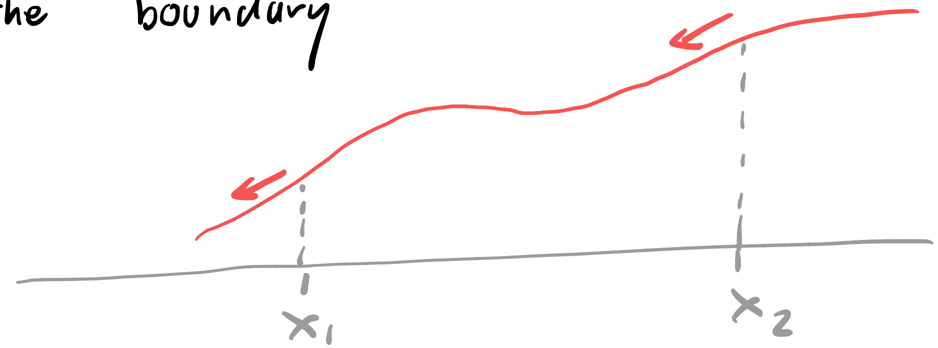
No observed change in the heat distribution

Observed change in the distribution!

Heat decreases $[\partial_t u < 0]$ at local maxima $[\partial^2_{xx} u < 0]$

Heat increases $[\partial_t u > 0]$ at local minima $[\partial^2_{xx} u > 0]$

The amount of heat in a subinterval $[x_1, x_2]$ changes via in / out flow at the boundary



$$\partial_t \underbrace{\int_{x_1}^{x_2} u(x,t)\,dx}_{\text{Heat contained in } [x_1,x_2] \text{ at time } t} = \partial_x u(x_2,t) - \partial_x u(x_1,t) = \int_{x_1}^{x_2} \partial_{xx} u(x,t)\,dx$$

influx at x_2 proportional to $\partial_x u(x_2,t)$

outflux at x_1 proportional to $\partial_x u(x_1,t)$

$$\Longrightarrow \quad \partial_t u(x,t) = \partial^2_{xx} u(x,t)$$

Heurestically, without source terms

More generally, we can study variants of the heat equation such as

$$\partial_t u = \kappa \, \partial_{xx}^2 u - \alpha u + f$$

material parameter
$\kappa > 0$
heat conductivity

material parameter
$\alpha \geq 0$
dampening parameter

VIII. 2 Solution via Fourier series

We search $u : [0, L] \times \mathbb{R}_0^+ \to \mathbb{R}$ satisfying

$$\partial_t u = \partial_{xx} u + f \qquad 0 < x < L, \quad t > 0$$

with initial data

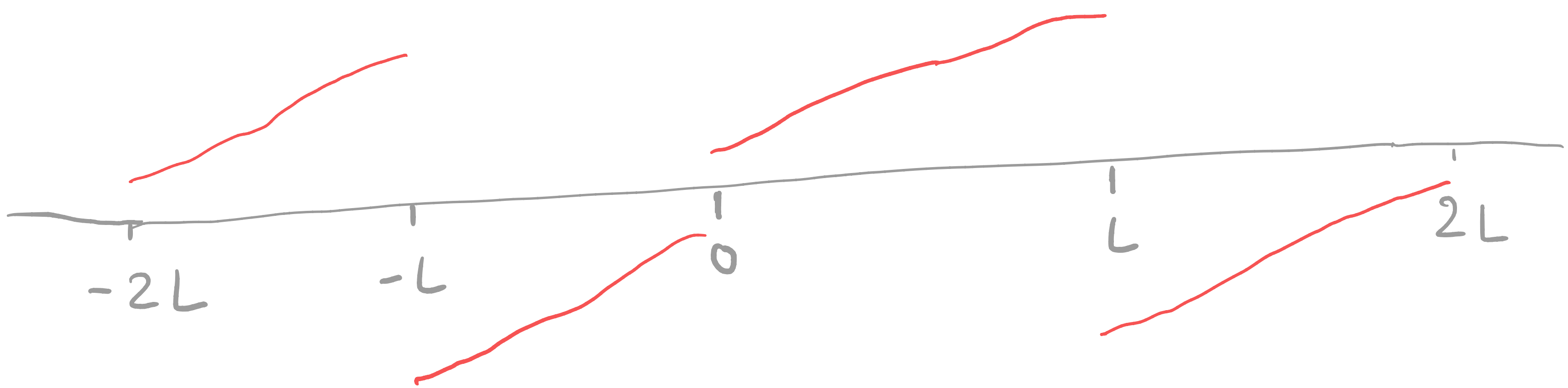
$$u(x, 0) = u_0(x), \qquad 0 < x < L$$

and homogenous Dirichlet boundary conditions

$$u(0, t) = 0, \qquad u(L, t) = 0, \qquad t > 0$$

General idea:
- Decompose u into Fourier modes
- Solve a Cauchy problem in time for each Fourier mode
- Put this back together to find the solution

1. We extend u_0 and f to odd functions over $\mathbb{R}$ with period $2L$



The Fourier series of odd periodic functions consists only of Fourier sine modes

The Fourier coefficients of an odd function with period $T > 0$ are

$$b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{2\pi n}{T} x\right) dx \qquad n = 1, 2, \ldots$$

The coefficients $a_n = 0$ all vanish.

In this particular setting

$$u_0(x) = \sum_{n=1}^{\infty} b_n^0 \sin\left(\frac{\cancel{2}\pi n}{\cancel{2}L} x \right)$$

$$= \sum_{n=1}^{\infty} b_n^0 \sin\left(\frac{\pi n}{L} x \right)$$

$$b_n^0 = \frac{2}{L} \int_0^L u_0(x) \sin\left(\frac{\pi n}{L} x \right)$$

using $T = 2L$ and u is odd

The Fourier series for f will be given for each time $t > 0$

$$f(x,t) = \sum_{n=1}^{\infty} \beta_n(t) \sin\left(\frac{\pi n}{L} x \right)$$

$$\beta_n(t) = \frac{2}{L} \int_0^L f(x,t) \sin\left(\frac{\pi n}{L} x \right) dx$$

Here, for each fixed $t > 0$, we take the Fourier series of $f(x,t)$, and the coefficients $\beta_n(t)$ depend on t.

We assume that u has a Fourier sine series

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t) \sin\left(\frac{\pi n}{L} x\right)$$

Notice: $u(x,t)$ already satisfies the boundary conditions

$$u(0,t) = 0, \qquad u(L,t) = 0$$

because u is a Fourier sine series with period $2L$. That makes sense if u is odd with period $2L$, and that follows because we assume u_0 and f to be odd with period $2L$.

2. We construct the Cauchy problems for the Fourier coefficients of u.

Goal: Find $b_n(t)$ **Need:** Differential Equation