06.03.2025

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Recap : Complex derivative

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

$$= \partial_x u(x,y) + \partial_x v(x,y) \cdot i$$

Complex derivative exists if and only if Cauchy-Riemann equations hold

$$\partial_x u = \partial_y v, \qquad \partial_y u = -\partial_x v$$

Functions with complex derivatives:

- polynomials: $z^3 + z^2$
- negative powers: z^{-1}, z^{-2}
- rational functions

$\Big\}$ except at poles

- exponential function e^z
- logarithm $\log(z) = \log|z| + \arg(z) \cdot i$

NB: The imaginary component of the logarithm is an arbitrary choice, we choose the principal argument of z.
This corresponds to the fact that $\exp(z) = \exp(z + 2\pi i \cdot k)$
We also say that we pick a "branch" of the complex logarithm.

- power functions: $z^\gamma := e^{\gamma \cdot \log(z)}$

With the logarithm, we define the power function with some complex exponent $\gamma \in \mathbb{C}$

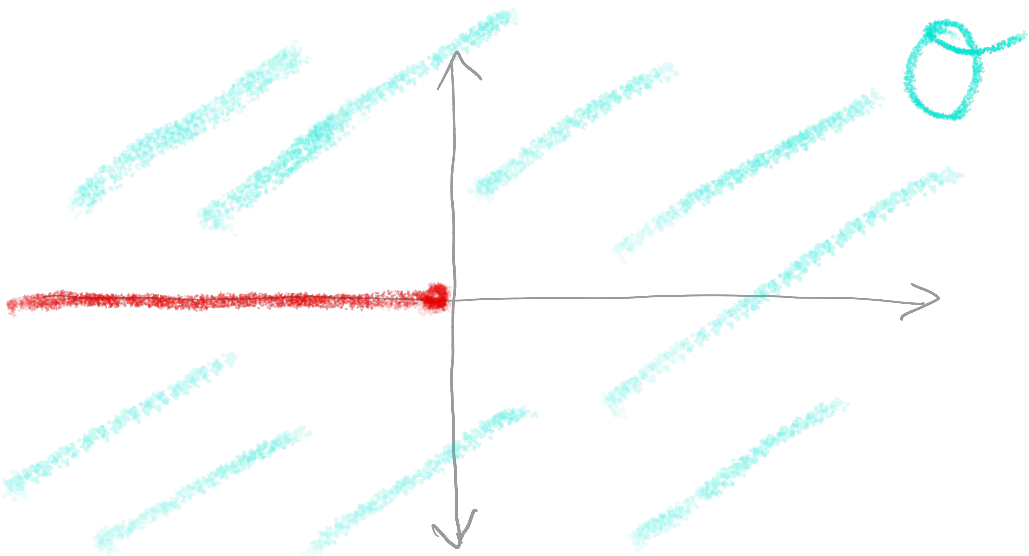
$$f(z) = z^\gamma = e^{\gamma \cdot \log(z)}$$

$$\left[z^\gamma = e^{\log(z^\gamma)} = e^{\gamma \cdot \log(z)} \right]$$

$$= e^{\gamma \log|z| + \gamma \arg(z) \cdot i}$$

This function is holomorphic over the same domain as the complex logarithm

$$\mathcal{O} = \mathbb{C} \setminus \{ z \in \mathbb{C} \mid \operatorname{Im}(z) = 0, \ \operatorname{Re}(z) \leq 0 \}$$



II Complex integration

We recall from vector analysis: let

$$\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2, \qquad \vec{F}(x_1, x_2) = \left(\vec{F}_1(x_1, x_2), \ \vec{F}_2(x_1, x_2) \right)$$

Given a curve $\gamma : [a,b] \to \mathbb{R}^2$, we have

$$\int_\gamma \vec{F} \, d\ell := \int_a^b \vec{F}(\gamma(t)) \bullet \gamma'(t) \, dt$$

produit scalaire

$$= \int_a^b F_1(\gamma(t)) \cdot \gamma_1'(t) + F_2(\gamma(t)) \cdot \gamma_2'(t) \, dt$$

With that in mind, we define the integral of a function $f : \mathbb{C} \to \mathbb{C}$ along a curve in $\mathbb{C}$.

II.1. Definition

Let $[a,b] \subseteq \mathbb{R}$ be a closed interval, $a \leq b$, and suppose we have a differentiable curve $\gamma : [a,b] \to \mathbb{C}$.

Let Γ be the image of the curve γ.

If $f : \Gamma \to \mathbb{C}$ is a continuous function, then

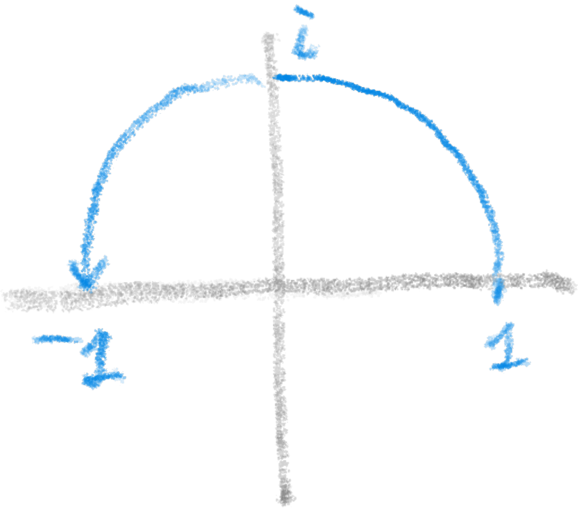
$$\int_\Gamma f(z)\,dz := \int_\gamma f(z)\,dz := \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt$$

complex multiplication

defines the integral of f along the curve Γ.

NB: Since γ is differentiable, it has derivative $\gamma' : (a,b) \to \mathbb{C}$

Example 1 Let Γ be the upper half-circle around 0 with radius 1



$$\gamma : [0, \pi] \to \mathbb{C}, \quad \gamma(t) = e^{it}$$

$$\gamma'(t) = i \cdot e^{it}$$

We integrate $f(z) = z^2$

$$\int_\Gamma z^2 \, dz = \int_0^\pi (e^{it})^2 \cdot i e^{it} \, dt = i \int_0^\pi e^{3it} \, dt$$

$$(\ast) = i \left[\frac{e^{3it}}{3i} \right]_{t=0}^{t=\pi} = \frac{1}{3} \left[e^{3it} \right]_{t=0}^{t=\pi}$$

$$= \frac{1}{3} \left[(-1) - (1) \right] = -2/3$$

$(\ast)$ Here, we use the fundamental theorem of calculus for the function $(a, b) \to \mathbb{C}, \ t \to e^{3it}$

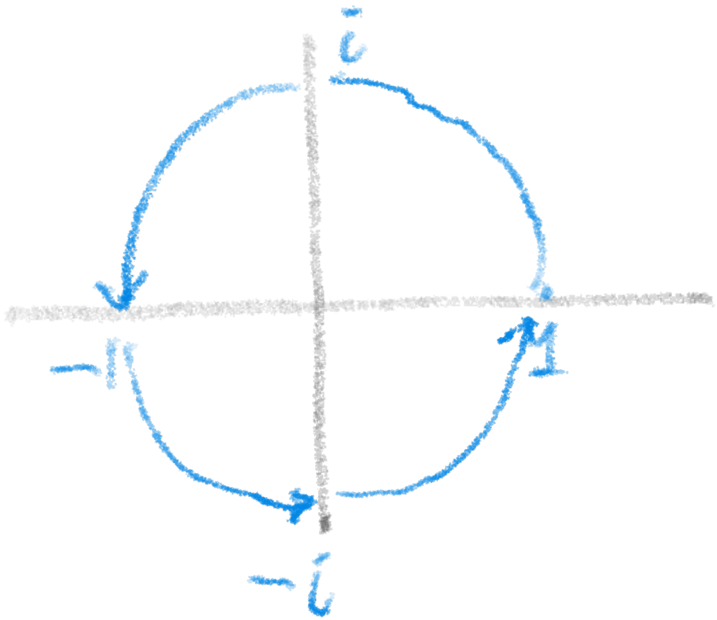
Example 2

Let Γ be the unit circle.

$$\gamma : [0, 2\pi] \to \mathbb{C}, \qquad \gamma(t) = e^{it}$$

Again, we integrate $f(z) = z^2$:

$$\int_\Gamma z^2 \, dz = \int_0^{2\pi} (e^{it})^2 \, i e^{it} \, dt = \ldots = \frac{1}{3} \left[e^{it} \right]_{t=0}^{t=2\pi} = 0$$



Q: Is the curve integral of a function $f : \mathbb{C} \to \mathbb{C}$ along a closed differentiable curve always zero?

A: No!

Example 3 Same Γ and γ but $f(z) = 1/z$

$$\int_\Gamma 1/z \, dz = \int_0^{2\pi} \frac{1}{e^{it}} \cdot i e^{it} \, dt = \int_0^{2\pi} i \, dz = 2\pi \cdot i$$

Example 4 Same Γ and γ but $f(z) = 1/z^2$

$$\int_\Gamma 1/z^2 \, dz = \int_0^{2\pi} \frac{i e^{it}}{(e^{it})^2} = i \int_0^{2\pi} e^{-it} \, dt$$

$$= i \left[\frac{e^{-it}}{-i} \right]_{t=0}^{t=2\pi} = 0$$

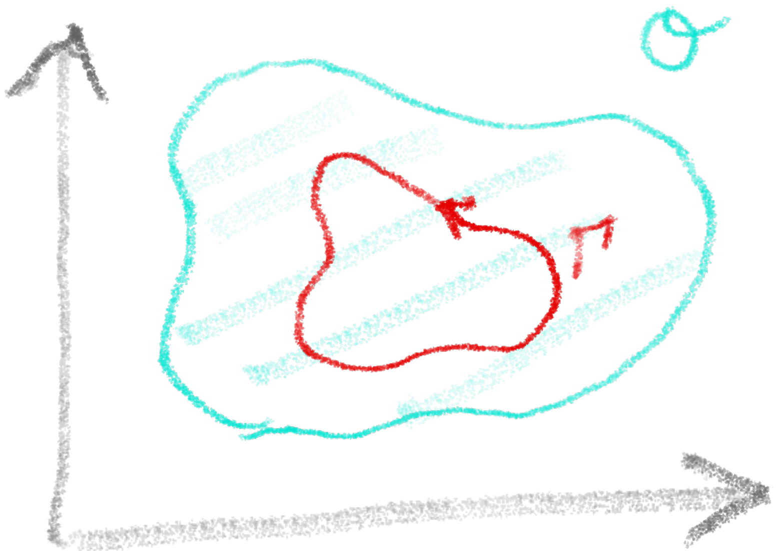
Remark: This definition of curve integral extends to curves that are piecewise differentiable (e.g. squares, combinations of lines and semicircles)

II.2 Cauchy theorem

Let $\mathcal{O} \subseteq \mathbb{C}$ be an open simply-connected set and let $\Gamma \subseteq \mathcal{O}$ be a closed (piecewise) differentiable curve.

Let $f : \mathcal{O} \to \mathbb{C}$ be holomorphic over $\mathcal{O}$. Then

$$\int_\Gamma f(z)\,dz = 0$$



1) For practical computations, we are given f and Γ, and we pick a suitable $\mathcal{O}$ to verify the conditions of the theorem

2) often, $\mathcal{O} = \mathbb{C}$ or $\mathcal{O}$ = unit disk

3) "simply-connected" means in practice that $\mathcal{O}$ has only one part and no holes

Examples 1 Let Γ be the unit circle and $f(z) = z^2$.

We apply the Cauchy theorem with $\mathcal{O} = \mathbb{C}$ and get

$$\int_\Gamma z^2 \, dz = 0$$

Example 2: Let Γ be the unit circle and $f(z) = 1/z$

Here, $f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$ is not defined at zero.

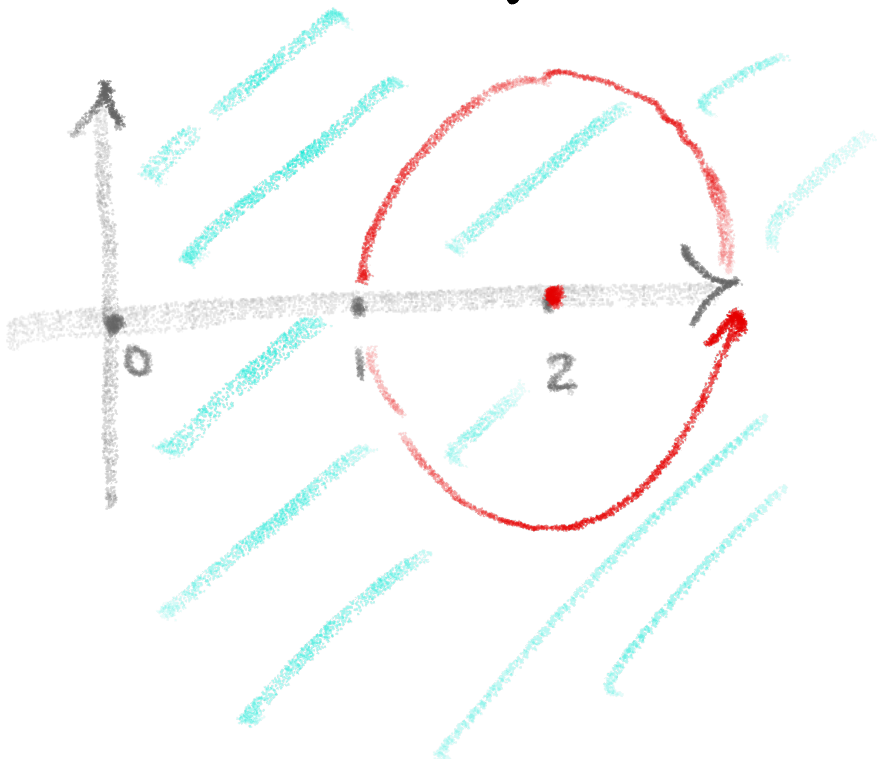
$$\int_\Gamma 1/z \, dz = 2\pi \cdot i \neq 0$$

Indeed, we cannot apply the Cauchy theorem! We cannot find a suitable $\mathcal{O}$. Any open set $\mathcal{O} \subseteq \mathbb{C}$ over which f is holomorphic and that contains the unit circle must leave $z_0 = 0$. Hence $\mathcal{O}$ must have a hole and can't be simply-connected.

Example 3

Let $f: \mathbb{C} \setminus \{0\} \to \mathbb{C}$, $f(z) = 1/z$ and

$\gamma: [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = 2 + e^{it}$



We can apply the Cauchy theorem

$$\int_\gamma f(z)\, dz = 0$$

Possible choice:

$$\mathcal{O} := \{ z \in \mathbb{C} \mid \operatorname{Re} z > 0 \}$$

Alternative choices

$$\mathcal{O} := \{ z \in \mathbb{C} \mid |z - 2| < 1.1 \}$$

We verify this result explicitly:

$$\int_\Gamma 1/z \, dz = \int_0^{2\pi} \frac{1}{2+e^{it}} ie^{it} \, dt = \int_0^{2\pi} \frac{ie^{it}}{2+e^{it}} \, dt$$

$$= \int_0^{2\pi} \log(2+e^{it})' \, dt$$

$$= \left[\log(2+e^{it}) \right]_{t=0}^{t=2\pi} = (\log 3 - \log 3) = 0$$

NB: This works as expected because the curve avoids the jump of the logarithm along the negative numbers

Proof of the Cauchy theorem (outline): Let $\gamma(t) = \alpha(t) + \beta(t) i$. Then

$$\int_\gamma f(z)\,dz = \int_a^b \big(u(\gamma(t)) + v(\gamma(t))\, i \big) \cdot \big(\alpha'(t) + \beta'(t)\, i \big)\, dt$$

$$= \int_a^b u(\gamma(t))\alpha'(t) - v(\gamma(t))\beta'(t)\, dt + \int_a^b v(\gamma(t))\alpha'(t) + u(\gamma(t))\beta'(t)\, dt \cdot i$$

$$= \int_a^b \underbrace{\begin{pmatrix} u(\gamma(t)) \\ -v(\gamma(t)) \end{pmatrix}}_{\vec{F}(\gamma(t))} \cdot \underbrace{\begin{pmatrix} \alpha'(t) \\ \beta'(t) \end{pmatrix}}_{\gamma'(t)} dt + \int_a^b \underbrace{\begin{pmatrix} v(\gamma(t)) \\ u(\gamma(t)) \end{pmatrix}}_{\vec{G}(\gamma(t))} \cdot \underbrace{\begin{pmatrix} \alpha'(t) \\ \beta'(t) \end{pmatrix}}_{\gamma'(t)} dt \cdot i$$

$$= \int_\Gamma \vec{F}\, dl + \int_\Gamma \vec{G}\, dl \cdot i$$

Here, we interpret $\mathbb{C}$ as the plane $\mathbb{R}^2$

$$= \iint_{\text{int}\,\Gamma} \underbrace{\text{curl}\,\vec{F}}_{-\partial_x v - \partial_y u = 0} + \iint_{\text{int}\,\Gamma} \underbrace{\text{curl}\,\vec{G}}_{\partial_x u - \partial_y v = 0} = 0$$

$\Leftarrow$ Cauchy-Riemann Equations

II.3 An extension of the Cauchy theorem