Example 3 Consider $f : \mathbb{C} \setminus \{0,1\} \to \mathbb{C}$ with

$$f(z) = \frac{2}{z} + \frac{3}{z-1} + \frac{1}{z^2}$$

$$= \frac{3}{z-1} + \frac{1+2z}{z^2}$$

f has pole of order 1 at $z_0 = 1$, and another pole of order 2 at $z_0 = 0$

$$\mathrm{Res}_1(f) = \lim_{z \to 1} (z-1) f(z)$$

$$= \lim_{z \to 1} \left(3 + (z-1) \frac{1+2z}{z^2} \right)$$

$$= 3 + 0 \cdot \frac{1+2 \cdot 1}{1} = 3$$

$$\operatorname{Res}_0(f) = \lim_{z \to 0} \frac{d}{dz}\left[(z-0)^2 f(z) \right]$$

$$= \lim_{z \to 0} \frac{d}{dz}\left[\frac{3z^2}{z-1} + 1 + 2z \right]$$

$$= \lim_{z \to 0} \left[\frac{6z(z-1) - 3z^2}{(z-1)^2} + 2 \right] = 0 + 2 = 2$$

That provides the residues at the poles.

[Any integral $\int_\gamma f(z)\,dz$ along some closed (piecewise) differentiable curve can be computed with the residue theorem.]

Let us suppose γ is a single piecewise differentiable curve.
We make a case distinction:

Case a) $0, 1 \notin \overline{\operatorname{int}\gamma}$ $\underset{\substack{\text{Residue thm} \\ \text{Cauchy thm}}}{\Longrightarrow}$ $\int_\gamma f(z)\,dz = 0$

Case b) $0 \in \operatorname{int}\gamma$, $1 \notin \overline{\operatorname{int}\gamma}$ $\Rightarrow$

$$\int_\gamma f(z)\,dz = 2\pi i \cdot \operatorname{Res}_0(f) = 2\pi i \cdot 2 = 4\pi i$$

Case c) $0 \notin \overline{\operatorname{int}\gamma}$, $1 \in \operatorname{int}\gamma$ $\Rightarrow$

$$\int_\gamma f(z)\,dz = 2\pi i \cdot \operatorname{Res}_1(f) = 2\pi i \cdot 3 = 6\pi i$$

Case d) $0, 1 \in \operatorname{int}\gamma$ $\Rightarrow$

$$\int_\gamma f(z)\,dz = 2\pi i \left(\operatorname{Res}_0(f) + \operatorname{Res}_1(f) \right) = 10\pi i$$

Case e) If $0 \in \gamma$ or $1 \in \gamma$, then the integral is not defined.

Example 4 The function

$$f(z) = \exp(1/z) = 1 + 1z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3!} + \frac{z^{-4}}{4!} + \dots$$

has an essential singularity at $z_0 = 0$. The residue there is

$$\mathrm{Res}_0(f) = 1 = c_{-1}$$

as seen from the Laurent series.

Example 5 The function

$$f(z) = \exp(1/z^2) = 1 + \frac{z^{-2}}{1!} + \frac{z^{-4}}{2!} + \frac{z^{-6}}{3!} + \dots$$

has an essential singularity at $z_0 = 0$, whose residue equals

$$\mathrm{Res}_0(f) = 0.$$

IV. 2 Applications to Integrals

Example 1 We want to calculate

$$\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\, dx$$

Towards that end, we study the function $f(z) = \frac{1}{z^2+1}$

We introduce the linear segment $L_r : [-r, r] \to \mathbb{C}, \quad t \mapsto t$
Then

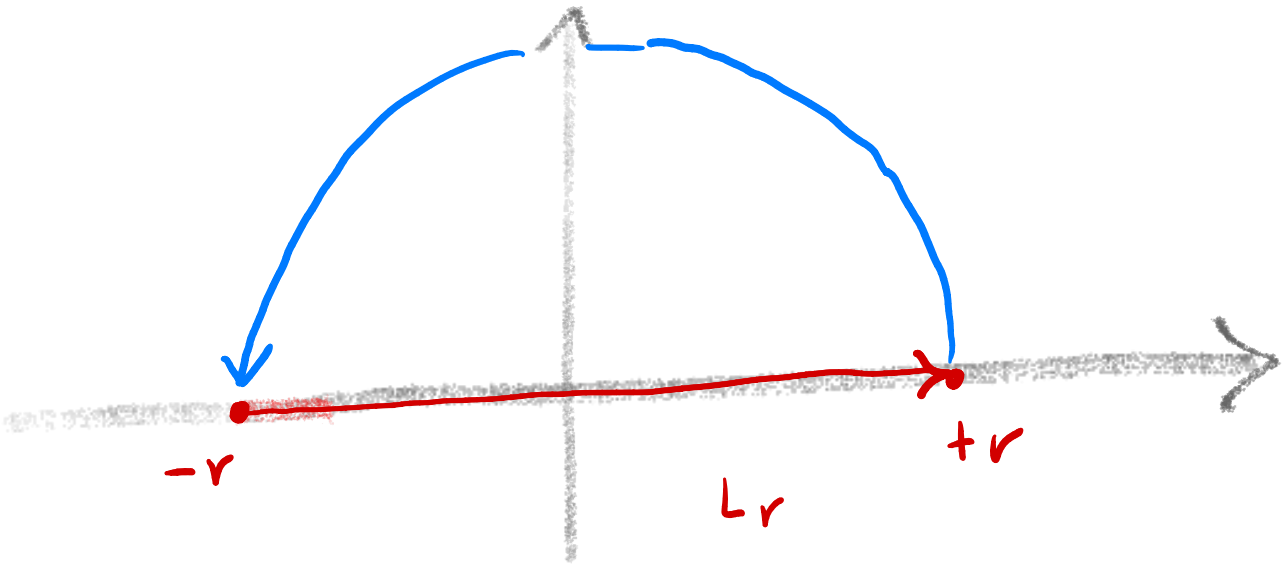
$$\int_{L_r} f(z)\, dz = \int_{-r}^{r} \frac{1}{z^2+1}\, dz \xrightarrow[r \to \infty]{} \int_{-\infty}^{+\infty} \frac{1}{z^2+1}\, dz$$

We make a detour through the complex plane, introducing the half-arc

$$C_r : [0, \pi] \to \mathbb{C}, \qquad t \to r \cdot e^{it}$$

of radius r centered at 0.

We compute the line integral of f along $\Gamma_r := L_r \cup C_r$



Γ_r is a closed piecewise differentiable curve.
We apply the residue theorem.

$$\int_{\Gamma_r} f(z)\,dz = \int_{\Gamma_r} \frac{1}{z^2+1}\,dz = \int_{\Gamma_r} \frac{1}{(z+i)(z-i)}\,dz$$

By the residue theorem, for very large $r > 0$

$$\int_{\Gamma_r} f(z)\,dz = 2\pi i \cdot \operatorname{Res}_i(f)$$

We compute the residue:

$$\operatorname{Res}_i(f) = \lim_{z \to i} (z-i) f(z) = \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}$$

Hence

$$\int_{\Gamma_r} f(z)\,dz = \pi$$

[Exercise: $\operatorname{Res}_{-i}(f) = \dots$, not needed here]

Here, we have used:

- Γ_r encloses only the pole at i, not the one at $-i$, and so only the residue at i matters for the residue theorem
- The pole at i is of first order, which determines the formula for the computing the residue.

For our original goal, we observe

$$\pi = \int_{\Gamma_r} f(z)\,dz = \int_{L_r} f(z)\,dz + \int_{C_r} f(z)\,dz$$

We want to show this goes to zero as $r \to \infty$

As $r \to \infty$, we already know

$$\int_{L_r} f(z)\,dz \longrightarrow \int_{-\infty}^{+\infty} f(z)\,dz$$

What about the integral along C_r in the limit?

$$\int_{C_r} f(z)\, dz = \int_0^{\pi} \frac{1}{r^2 e^{2it} + 1} \cdot rie^{it}\, dt$$

We bound the absolute value of the integral:

$$\left| \int_{C_r} f(z)\, dz \right| \leq \int_0^{\pi} \frac{1}{|r^2 e^{2it} + 1|} |rie^{it}|\, dt$$

We have

$$|rie^{it}| = r \cdot |ie^{it}| = r \cdot \underbrace{|i|}_{=1} \cdot \underbrace{|e^{it}|}_{=1} = r$$

and for r large, we have approximately

$$|r^2 e^{2it} + 1| \approx |r^2 e^{2it}| = r^2.$$

Hence, for large r we find:

$$\left| \int_{C_r} f(z)\,dz \right| \approx \int_0^{\pi} \frac{r}{r^2} = \frac{\pi}{r} \xrightarrow[r \to \infty]{} 0$$

As r goes to ∞, the absolute value of the integral along C_r goes to zero, and so the value of the integral itself must go to zero.

In summary

$$\pi = \int_{\Gamma_r} f(z)\,dz = \int_{L_r} f(z)\,dz + \int_{C_r} f(z)\,dz$$

$$\xrightarrow[r \to \infty]{} \int_{-\infty}^{+\infty} \frac{1}{z^2+1}\,dz + 0$$

Hence

$$\boxed{\int_{-\infty}^{+\infty} \frac{1}{x^2+1}\,dx = \pi}$$

Example 2 Consider the indefinite integral

$$\int_{-\infty}^{+\infty} R(x) \cdot e^{i\alpha x} \, dx, \qquad \text{where} \quad \alpha \geqslant 0,$$

and where $R(x) = P(x) / Q(x)$ is a rational function such that P and Q are polynomials with

- $\deg Q \geqslant \deg P + 2$
- $Q(x) \neq 0$ for all $x \in \mathbb{R}$

Previous example is a special case

$P(x) = 1, \qquad Q(x) = x^2 + 1, \qquad \alpha = 0$

Since Q has at most $\deg Q$ zeroes throughout $\mathbb{C}$, the function $R(z) = P(z) / Q(z)$ has at most $\deg Q$ singularities.

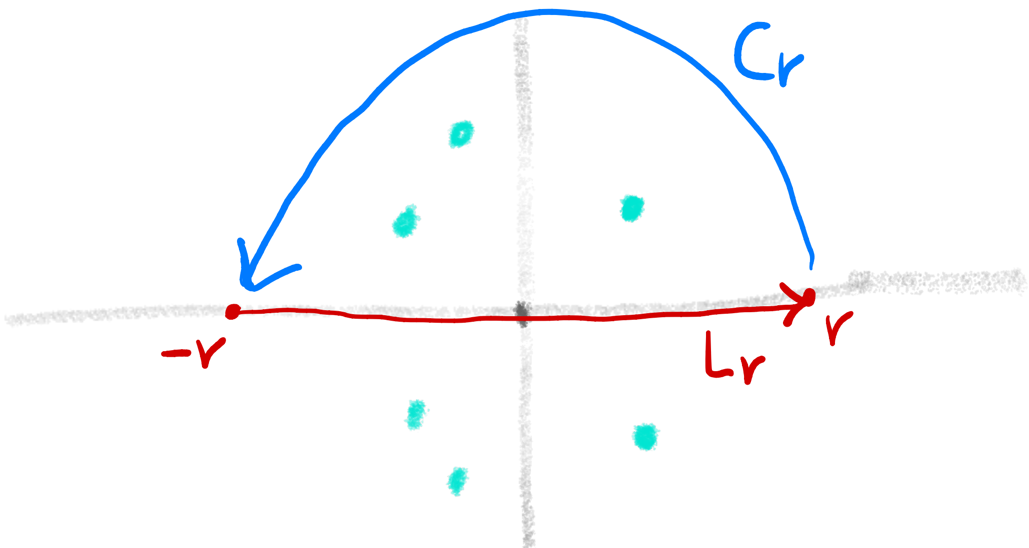
[Fundamental theorem of algebra]

Let $r > 0$ be a radius large enough such that $B_r(0)$ contains all singular points of $R(z)$. We introduce

$$L_r = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) = 0, \quad -r \leq \mathrm{Re}(z) \leq r \}$$

$$C_r = \{ z \in \mathbb{C} \mid z = r e^{i\theta}, \quad 0 \leq \theta \leq \pi \}$$

We join both curves to a closed curve Γ_r.



The function $R(z)\, e^{i\alpha z}$ has singularities in the upper half-plane:

$$z_1, \; z_2, \; \dots, \; z_m$$

For r large enough, Γ_r encircles those singularities.

$$\int_{\Gamma_r} R(z)\, e^{i\alpha z}\, dz = 2\pi i \sum_{j=1}^{m} \mathrm{Res}_{z_j}\left(R(z) e^{i\alpha z} \right)$$

In addition,

$$\int_{\Gamma_r} R(z)\, e^{i\alpha z}\, dz = \int_{-r}^{+r} R(x)\, e^{i\alpha x} dx + \int_{C_r} R(z)\, e^{i\alpha z}\, dz$$

In the limit

$$\int_{-r}^{+r} R(x)\, e^{i\alpha x} \xrightarrow[r\to\infty]{} \int_{-\infty}^{+\infty} R(x)\, e^{i\alpha x}\, dx$$

It remains to control the integral over C_r : it is sufficient to show that the absolute value goes to zero as $r \to \infty$.

$$\left| \int_{C_r} R(z)\, e^{i\alpha z}\, dz \right| = \left| \int_0^{\pi} \frac{P(re^{it})}{Q(re^{it})}\, e^{i\alpha r e^{it}} \cdot ire^{it}\, dt \right|$$

$$\leq \int_0^{\pi} \left|\frac{P(re^{it})}{Q(re^{it})}\right| \cdot \left|e^{i\alpha re^{it}}\right| \cdot \underbrace{|i|}_{=1} \cdot r \cdot \underbrace{|e^{it}|}_{=1} \, dt$$

We use

$$e^{i\alpha r e^{it}} = e^{i\alpha r(\cos(t) + i\sin(t))}$$

$$= e^{i\alpha r\cos(t) - \alpha r\sin(t)}$$

$$= \underbrace{e^{i\alpha r\cos(t)}}_{\text{on the complex unit circle}} \cdot \underbrace{e^{-\alpha r\sin(t)}}_{\leq 1 \text{ because } \alpha \geq 0 \text{ and } \sin(t) \geq 0 \text{ for } 0 \leq t \leq \pi}$$

So

$$\left|e^{i\alpha re^{it}}\right| \leq 1.$$

Finally,

$$\left|\frac{P(re^{it})}{Q(re^{it})}\right| \cdot r$$ behaves proportionally like

$$\frac{1}{r^2} \cdot r = \frac{1}{r}$$ for r large

This is where we use $\deg Q \geqslant \deg P + 2$.

In summary,

$$\left| \int_{C_r} f(z)\,dz \right| \xrightarrow[r \to \infty]{} \approx \pi \cdot \frac{1}{r}$$

which vanishes as r goes to infinity. Final result

$$\boxed{\int_{-\infty}^{+\infty} R(z)\,e^{i\alpha z}\,dz = 2\pi i \sum_{j=1}^{m} \operatorname{Res}_{z_j}\left(R(z)\,e^{i\alpha z}\right)}$$

Example 3 We can compute the integral

$$\int_{-\infty}^{+\infty} \frac{x^2}{16+x^4}\,dx = \int_{-\infty}^{+\infty} \frac{x^2}{16+x^4}\, e^{i0x}\,dx$$

Here $R(z) = P(z)/Q(z)$ with $P(z) = z^2$, $Q(z) = 16 + z^4$.