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VII. 1. First-order ordinary differential equations

Let $a : [0, \infty) \to \mathbb{R}$ and $f : [0, \infty) \to \mathbb{R}$ be piecewise continuous functions. Let $y_0 \in \mathbb{R}$.

We study the initial value problem

$$
\begin{aligned}
y'(t) + a(t)\, y(t) &= f(t), \qquad t > 0 \\
y(0) &= y_0
\end{aligned}
$$

This also known as 1st order Cauchy problem.

Typical application: exponential growth / decay over time $t > 0$

y_0 is the initial value at time $t = 0$.

$a(t)$ is the rate of growth / decay at time t

$f(t)$ is the source term at time t

Example: $y(t)$ is the population size of bacteria, starting with size y_0 at time $t = 0$. The source term $f(t)$ denotes additional influx / outflux of bacteria.

$$y'(t) = f(t) - a(t)\, y(t)$$

The population grows / shrinks by itself proportionally to its size if $a(t) < 0$ / $a(t) > 0$.

Example: Radioactive decay, financial interests

To solve the initial value problem

$$\begin{aligned} y'(t) + a(t)\,y(t) &= f(t), \qquad t > 0 \\ y(0) &= y_0 \end{aligned}$$

We use the Laplace transform.

For simplicity, we assume $a(t) = a \in \mathbb{R}$.

Applying the Laplace transform yields:

$$z\,Y(z) - y(0) + a \cdot Y(z) = F(z)$$

We use $y(0) = y_0$ and isolate $Y(z)$:

$$(z + a)\,Y(z) = F(z) + y_0$$

$$Y(z) = \frac{F(z) + y_0}{z + a}$$

Having isolated $Y(z)$, we try to invert the Laplace transform:

$$Y(z) = \frac{1}{z+a} F(z) + y_0 \frac{1}{z+a}$$

We know that

$$\frac{1}{z+a} = \mathcal{L}[e^{-at}]$$

Hence,

$$Y(z) = \mathcal{L}[e^{-at}]\, \mathcal{L}[f] + y_0\, \mathcal{L}[e^{-at}]$$

We use the formula for the Laplace transform of convolutions:

$$y(t) = \int_0^t f(s)\, e^{-a(t-s)}\, ds + y_0\, e^{-at}$$

This solves the initial value problem:

we manually check the derivatives and the initial value.

Clearly, for $t = 0$, we have $y(0) = y_0$. Moreover,

$$\frac{d}{dt} y(t) = \underbrace{f(t) + \int_0^t f(s)\,(-a)\, e^{-a(t-s)}\, ds}_{\text{Leibniz integral formula}} + (-a) y_0 e^{-at}$$

$$= f(t) - a \underbrace{\left(\int_0^t f(s)\, e^{-a(t-s)}\, ds + y_0 e^{-at} \right)}_{= y(t)}$$

$$\Rightarrow \quad y'(t) = f(t) - a\, y(t)$$

<u>Remark:</u> when the growth/decay coefficient $a(t)$ is not constant, then the argument is considerably more complicated.

Special case: $f(t) = 0$. Then $y'(t) = -a\,y(t)$ is solved by

$$y(t) = y_0 e^{-at}$$

Special case: $a(t) = 0$. Then $y'(t) = f(t)$ is solved by

$$y(t) = \int_0^t f(s)\,ds + y_0$$

First-order initial value problems need initial data for the function itself. Second-order initial value problems need initial data for the function and its first derivative.

VII. 2. Second-order ordinary differential equations

We consider the initial value problem: given $\omega, \alpha \geq 0$, we search $y : [0, \infty) \to \mathbb{R}$ twice-differentiable such that

$$y''(t) = -\omega^2 y(t) - 2\alpha y'(t), \qquad t > 0$$

$$y(0) = y_0$$

$$y'(0) = v_0$$

Here, y_0 and v_0 are the initial position and velocity of the system and $y''(t)$ denotes the accelleration at time $t > 0$.

When $y(t)$ is large, then $-\omega^2 y(t)$ effects negative accelleration.

When $y'(t)$ is large, then $-2\alpha y'(t)$ will cause dampening

Example: Harmonic oscillator. Here $\omega, \alpha \geq 0$ are material parameters.

To solve this problem, we use the Laplace transform. This will involve a case distinction, depending on ω and α.

The Laplace transform of

$$y''(t) = -\omega^2 y(t) - 2\alpha y'(t)$$

is the following

$$z^2 Y(z) - z y(0) - y'(0) = -\omega^2 Y(z) - 2\alpha\left(z Y(z) - y(0)\right)$$

We use the initial data,

$$z^2 Y(z) - z y_0 - v_0 = -\omega^2 Y(z) - 2\alpha z Y(z) + 2\alpha y_0,$$

and isolate $Y(z)$:

$$(z^2 + \omega^2 + 2\alpha z)\, Y(z) = z y_0 + v_0 + 2\alpha y_0$$

$$Y(z) = y_0 \frac{z}{z^2 + \omega^2 + 2\alpha z} + \frac{2\alpha y_0 + v_0}{z^2 + \omega^2 + 2\alpha z}$$

$$= y_0 \frac{z}{(z+\alpha)^2 + \omega^2 - \alpha^2} + \frac{2\alpha y_0 + v_0}{(z+\alpha)^2 + \omega^2 - \alpha^2}$$

[completing the square : $z^2 + \omega^2 + 2\alpha z = z^2 + 2\alpha z + \alpha^2 + \omega^2 - \alpha^2$]

We conduct a case distinction in $\omega^2 - \alpha^2$.

<u>Case $\omega^2 - \alpha^2 = 0$</u>

$$Y(z) = y_0 \frac{z}{(z+\alpha)^2} + \frac{2\alpha y_0 + v_0}{(z+\alpha)^2}$$

$$= \frac{y_0 z + 2\alpha y_0 + v_0}{(z+\alpha)^2}$$

$$= y_0 \frac{(z+\alpha)}{(z+\alpha)^2} + \frac{\alpha y_0 + v_0}{(z+\alpha)^2}$$

$$= y_0 \frac{1}{z+\alpha} + \frac{\alpha y_0 + v_0}{(z+\alpha)^2}$$

We apply the inverse Laplace transform or use Laplace transform table:

$$\mathcal{L}[e^{-\alpha t}] = \frac{1}{z+\alpha}, \qquad \mathcal{L}[t e^{-\alpha t}] = \frac{1}{(z+\alpha)^2}$$

Therefore,

$$Y(z) = y_0 \mathcal{L}[e^{-\alpha t}] + (\alpha y_0 + v_0) \mathcal{L}[t e^{-\alpha t}]$$

$$\Rightarrow \quad y(t) = y_0 e^{-\alpha t} + (\alpha y_0 + v_0)\, t\, e^{-\alpha t}$$

$$= e^{-\alpha t} \left(y_0 + t(\alpha y_0 + v_0) \right)$$

This solves the ordinary differential equation.

Physically, this is the fastest non-oscillatory return to equilibrium.

If $v_0 = 0$, then the typical profile is:

Case $\omega^2 - \alpha^2 > 0$

We use the following formulas:

$$\mathcal{L}\left[e^{\beta t}\cos(\gamma t)\right] = \frac{z-\beta}{(z-\beta)^2+\gamma^2}$$

$$\mathcal{L}\left[e^{\beta t}\sin(\gamma t)\right] = \frac{\gamma}{(z-\beta)^2+\gamma^2}$$

With $\beta = -\alpha$ and $\gamma = \sqrt{\omega^2-\alpha^2}$, this becomes

$$\mathcal{L}\left[e^{-\alpha t}\cos\left(\sqrt{\omega^2-\alpha^2}\,t\right)\right] = \frac{z+\alpha}{(z+\alpha)^2+\omega^2-\alpha^2}$$

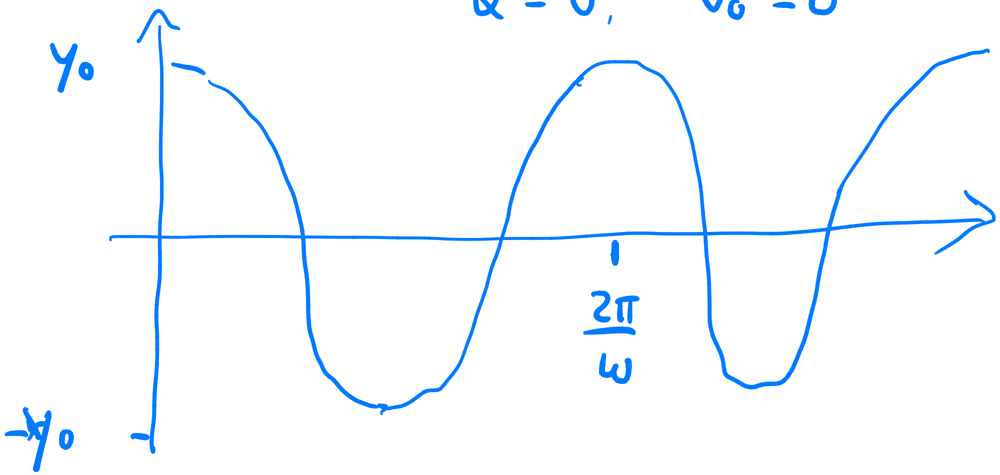
$$\mathcal{L}\left[e^{-\alpha t}\sin\left(\sqrt{\omega^2-\alpha^2}\,t\right)\right] = \frac{\sqrt{\omega^2-\alpha^2}}{(z+\alpha)^2+\omega^2-\alpha^2}$$

Now:

$$y(t) = y_0 e^{-\alpha t} \cos\left(\sqrt{\omega^2 - \alpha^2}\, t\right) + \frac{\alpha y_0 + v_0}{\sqrt{\omega^2 - \alpha^2}} e^{-\alpha t} \sin\left(\sqrt{\omega^2 - \alpha^2}\, t\right)$$

Physically, oscillatory decay

$\alpha = 0, \quad v_0 = 0$



$\Rightarrow \quad y(t) = y_0 \cos(\omega t)$

For positive $\alpha > 0$, this oscillation will decay to zero over time

Case $\omega^2 - \alpha^2 < 0$

$$Y(z) = y_0 \frac{z}{(z+\alpha)^2 - (\alpha^2 - \omega^2)} + \frac{2\alpha y_0 + v_0}{(z+\alpha)^2 - (\alpha^2 - \omega^2)}$$

Note $\alpha^2 - \omega^2 > 0$. We use the Laplace transforms:

For $\beta, \gamma \in \mathbb{R}$,

$$\mathcal{L}\left[e^{\beta t} \cosh(\gamma t) \right] = \frac{z - \beta}{(z-\beta)^2 - \gamma^2}$$

$$\mathcal{L}\left[e^{\beta t} \sinh(\gamma t) \right] = \frac{\gamma}{(z-\beta)^2 - \gamma^2}$$

We use $\beta = -\alpha$, $\gamma = \sqrt{\alpha^2 - \omega^2}$.

$$Y(z) = y_0 \mathcal{L}\left[e^{-\alpha t} \cosh\left(\sqrt{\alpha^2 - \omega^2}\, t \right) \right]$$

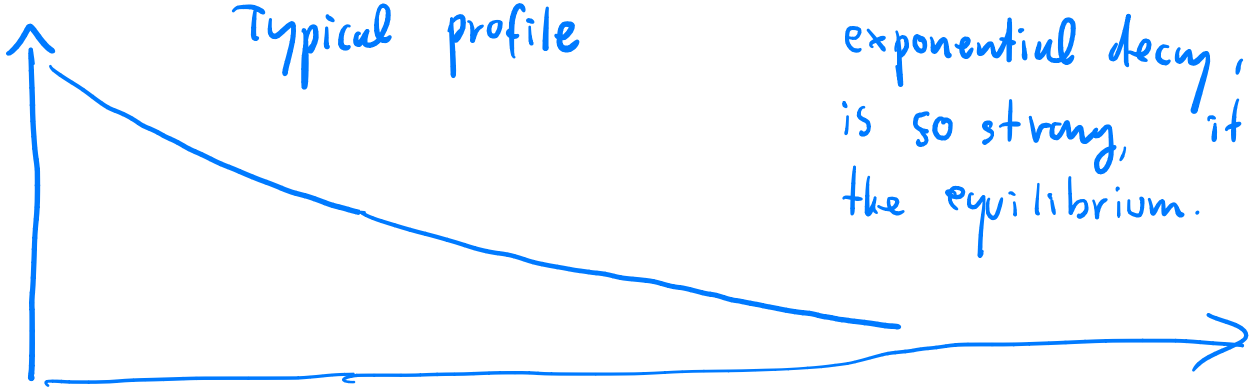
$$+ \frac{\alpha y_0 + v_0}{\sqrt{\alpha^2 - \omega^2}} \mathcal{L}\left[e^{-\alpha t} \sinh\left(\sqrt{\alpha^2 - \omega^2}\, t \right) \right]$$

$$\Rightarrow \quad y(t) = y_0 e^{-\alpha t} \cosh\left(\sqrt{\alpha^2 - \omega^2}\, t \right) + \frac{\alpha y_0 + v_0}{\sqrt{\alpha^2 - \omega^2}} e^{-\alpha t} \sinh\left(\sqrt{\alpha^2 - \omega^2}\, t \right)$$

Recall: $\cosh(x) = \dfrac{e^{-x} + e^{x}}{2}$, $\sinh(x) = \dfrac{e^{x} - e^{-x}}{2}$

(symmetric part of e^x) (odd part of e^x)

Typical profile



exponential decay, but the dampening $\alpha > 0$ is so strong, it slows down the decay towards the equilibrium.