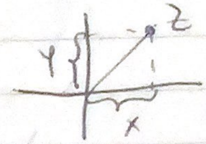
Today : chapter 3 : Holomorphic functions & Cauchy - Riemann equations

§ 1.0 Complex numbers : · cartesian & polar representations
· operations
· taking complex roots

- $i = \sqrt{-1}$, i.e $(i)^2 = -1$
- $z \in \mathbb{C}$ can be written as

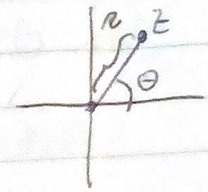
↳ $\boxed{z = x + iy}$ cartesian rep



$x := Re(z)$, $y := Im(z)$

↳ $\boxed{z = r e^{i\theta} \underset{\text{Euler's formula}}{=} r\cos\theta + i r \sin\theta}$

polar rep.



$r \in \mathbb{R}_{\geq 0}$
$\theta \in \mathbb{R}$

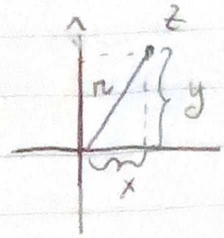
- We are introducing both because depending on what we have to do, one representation is more convenient than the other

How are the two related?

- Expressing x, y as a function of r and θ is easy

$$\boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

But we also want to express r and θ as functions of x and y



Pitagoras

$$\boxed{r = \sqrt{x^2+y^2} \;:= |z|}$$

absolute value of

norm the cpx number z

Rmk

let $\bar{z} = x - iy$ the complex conjugate of z

$\Rightarrow |z|^2 = z \cdot \bar{z}$

We also want to express

$\theta(z) =: \arg(z)$ argument of z as a function of x and y

Attention: $\arg(z)$ is a multivalued function

since a complex number is unchanged by circling around z

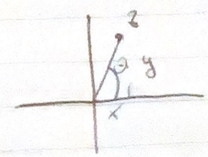
i.e For θ or $\theta + 2k\pi$ $k \in \mathbb{Z}$

sin and cos are the same!

Solution : restrict the interval of variation of arg(z) :

we consider $\arg(z) \in (-\pi, \pi]$

Now : we have the relation :

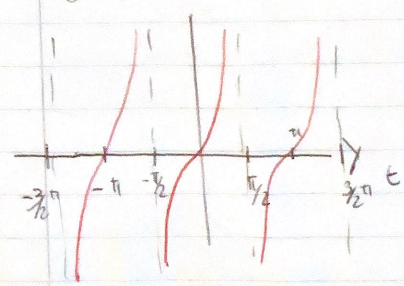


$$tg\,\theta = y/x$$

$\Rightarrow$ we want to define

$$\arg(z) = \operatorname{arctg}(y/x)$$

But : let us recall how the graph of $tg(t)$ looks like :

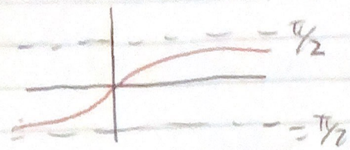


$\Rightarrow$ we can take arctg after restricting to one of the interval where tg in injective

But the value of the arctg depend on this choice.

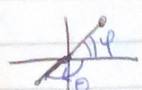
we take $t \in (-\pi/2, \pi/2)$

$\Rightarrow F^{-1}(t) = \operatorname{arctg}$

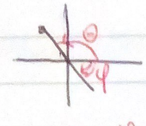


$$\Rightarrow \quad \underset{z \neq 0}{-\pi < \arg z \leq \pi} = \begin{cases} \operatorname{arctg} y/x & \text{if } x>0 \\ \pi/2 & x=0,\ y>0 \\ -\pi/2 & x=0\ \ y<0 \\ \pi + \operatorname{arctg}(y/x) & x<0,\ y>0 \\ -\pi + \operatorname{arctg}(y/x) & x<0,\ y<0 \end{cases}$$

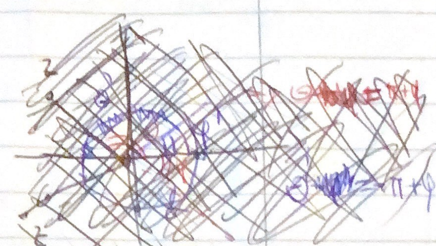
this function is what is called principal value of the argument



$\Theta = \varphi - \pi$



$\Theta = \varphi + \pi$



Operations

$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 y_2)$

$z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2)$

$z^{-1} = \frac{\bar{z}}{|z|}$

Given $w \in \mathbb{C}$, say we want to solve

$z^n = w \qquad n \in \mathbb{N}$

$\Leftrightarrow$ in polar coordinate $r^n e^{in\theta} = \rho e^{i\varphi}$

$\Rightarrow$ • $r = \sqrt[n]{\rho}$ well defined, $\rho \geq 0$

• $\theta = \frac{\varphi + 2k\pi}{n} \qquad k \in \mathbb{Z}$

Notice: For $k = 0, \ldots n-1$ we obtain different roots $\Rightarrow z^n = w$ has n distinct solutions!

Exercise: Draw in the cartesian plane the solution of

$z^5 = 1$, $z^4 = -4$, $z^3 = 2 + i$

§ 1.1. Complex & holomorphic functions

Def: A complex function is a function

$$f : \underbrace{\mathcal{O}}_{\text{open}} \subseteq \mathbb{C} \longrightarrow \mathbb{C}$$

$$z \longmapsto f(z)$$

$$z = x+iy, \qquad f(z) = \underbrace{u(x,y)}_{\operatorname{Re} f} + i\,\underbrace{v(x,y)}_{\operatorname{Im} f}$$

with $u, v : \mathbb{R}^2 \to \mathbb{R}$

Examples

- $f(z) = \bar{z} = \underbrace{x}_{x = u(x,y)} - i\,\underbrace{y}_{v(x,y) = -y}$

- $f(z) = z^2 = \underbrace{x^2 - y^2}_{u(x,y)} + i\,\underbrace{(2xy)}_{v(x,y)}$

- $f(z) = e^z = \underbrace{e^x \cos y}_{u(x,y)} + i\,\underbrace{e^x \sin y}_{v(x,y)}$

• complex logarithm

$$\log(z) = \log\|z\| + i \arg(z)$$

This is a multivalued function, because $\arg(z)$ is

We call principal value of the log the function obtained restricting $\arg(z) \in (-\pi, \pi]$

• $\cos z := \dfrac{e^{iz} + e^{-iz}}{2}$

• $\sin z = e^{iz} - e^{-iz}$

• $\cosh z = \dfrac{e^{z} + e^{-z}}{2}$

• $\sinh z = \dfrac{e^{z} - e^{-z}}{2}$

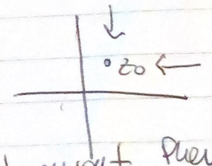
compute $u(x,y)$
$v(x,y)$

Definition: F is continuos if $\forall \varepsilon > 0, \exists \delta > 0$ s.t.

$$|z - z_0| < \delta \Rightarrow |F(z) - F(z_0)| < \varepsilon$$

In particular, F continuos

$$\Rightarrow \boxed{\lim_{z \to z_0} F(z) = F(z_0)}$$



independent then

the direction

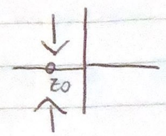
Example: The complex logarithm $\log z$

is not continus on

$$\{ z \in \mathbb{C} \mid \operatorname{Im}(z) = 0,\ \operatorname{Re}(z) \leq 0 \}$$

For $z=0$, not defined.

Indeed



$$\lim_{t \to 0^+} \log(x+it) = \log(|x|) + i\pi$$

$$\lim_{t \to 0^-} \log(x+it) = \log(|x|) - i\pi$$

Definition 8.1: Let $U \overset{\text{open}}{\subseteq} \mathbb{C}$, $f: U \to \mathbb{C}$

is holomorphic if $\forall z_0 \in U$ f

is differentiable at z_0, i.e.

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \quad \text{exists finite}$$

We call $f'(z_0)$ the limit

Theorem 8.2

Let $F : \Omega \to \mathbb{C}$ a complex function.

Denote $u(x,y) = \operatorname{Re} F(x+iy)$

$v(x,y) = \operatorname{Im} F(x+iy)$

The following are equivalent:

(1) F is holomorphic in Ω

(2) $u, v \in C^1(\Omega)$ and they satisfy the Cauchy-Riemann equations:

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

Furthermore

$$\boxed{F'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}}$$

Examples

- $F(z) = z^2 = (x^2 - y^2) + i\,2xy$ is holomorphic on $\mathbb{C}$

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial y}$$

- $f(z) = \bar{z} = x - iy$ is not holomorphic

$\frac{\partial u}{\partial x} = 1$, $\frac{\partial v}{\partial y} = -1$

- $f(z) = \frac{1}{z} = \frac{\bar{z}}{\|z\|^2} = \frac{x}{(x^2+y^2)} - i\frac{y}{(x^2+y^2)}$

is holomorphic in $U = \mathbb{C} \setminus 0$

$$\frac{\partial u}{\partial x} = \frac{y^2 - x^2}{(x^2+y^2)^2} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2} = -\frac{\partial v}{\partial x}$$

moreover, by the second part of the Thm

verify (exercise)

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -\frac{1}{z^2}$$

- $f(z) = e^z$ is holomorphic

and $f'(z) = e^z$

verify

- $\log z$ is holomorphic on

$$\mathcal{O} = \mathbb{C} \setminus \{z \in \mathbb{C} \mid \mathrm{Re}(z) \leq 0, \mathrm{Im}(z) = 0\}$$

and

$$(\log(z))' = \frac{1}{z}$$

To prove this recall the properties of the logarithm on the real numbers

+ fact that $(\operatorname{arctg}(t))' = \dfrac{1}{1+t^2}$

<u>Remark</u>: the property of the derivatives are the same as over $\mathbb{R}$:

- $(f+g)'(z_0) = f'(z_0) + g'(z_0)$
- $(f \cdot g)'(z_0) = f'(z_0) g(z_0) + f(z_0) g'(z_0)$
- $\left(\dfrac{f}{g}\right)'(z_0) = \dfrac{f'(z_0) g(z_0) - f(z_0) g'(z_0)}{g^2(z_0)}$
- $(f \circ g)' = f'(g(z_0)) \cdot g'(z_0)$

<u>Exercise</u>

Using this show that for $\gamma \in \mathbb{C}$

$f(z) = z^\gamma$ is holomorphic on $\mathcal{O} = \mathbb{C} \setminus \{\mathrm{Re}$

and $f(z)' = \gamma \cdot z^{\gamma-1}$ <u>hint</u> $= e^{\log(z^\gamma)}$