Exercices on distributions

Exercice 1. Let $H(t)$ be the piece-wise continuous function

$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

and define $u(t) = \cos(t)H(t)$. Verify that $u(t)$ is a solution of the following differential equation in $\mathcal{D}'_{\mathbb{R}}$ (i.e. distributional differential equation):

$$\left\langle D^2 u, \cdot \right\rangle + \langle u, \cdot \rangle = \langle D\delta_0, \cdot \rangle$$

where $\langle \delta_0, \cdot \rangle : \mathcal{D} \to \mathbb{R}$ is the Dirac mass distribution, defined by

$$\langle \delta_0, \varphi \rangle = \varphi(0)$$

HINTS: use the definition of distributional derivative; use integration by part; recall that the test functions φ and their derivatives are zero outside a bounded set.

Answer: We start by computing the second derivative of $u(t)$, where we use the fact that $(H(t))' = \delta_0(t)$:

$$\begin{aligned} u'(t) &= -\sin t \cdot H(t) + \cos t \cdot \delta_0(t) \\ u''(t) &= -\cos t \cdot H(t) - 2\sin t \cdot \delta_0(t) + \cos t \cdot \delta_0'(t) \end{aligned} \tag{1}$$

We know that

$$\int_{-\infty}^{\infty} \delta_0(t) f(t) \,\mathrm{dt} = f(0)$$

Also, using the integration by part we have

$$\int_{-\infty}^{\infty} \delta_0'(t) f(t) \,\mathrm{dt} = -\int_{-\infty}^{\infty} \delta_0(t) f'(t) \,\mathrm{dt} = -f'(0) \tag{2}$$

Hence, using (1)-(2) we can derive

$$\begin{aligned} \langle D^2 u, \phi \rangle + \langle u, \phi \rangle = &-\int_{-\infty}^{\infty} \cos t \cdot H(t)\phi(t) \,\mathrm{dt} - 2\int_{-\infty}^{\infty} \sin t \cdot \delta_0(t)\phi(t) \,\mathrm{dt} \\ &+ \int_{-\infty}^{\infty} \cos t \cdot \delta_0'(t)\phi(t) \,\mathrm{dt} + \int_{-\infty}^{\infty} \cos t \cdot H(t)\phi(t) \,\mathrm{dt} \\ &= -2\sin 0 \cdot \phi(0) - \cos 0 \phi'(0) + \sin 0 \phi(0) = -\phi'(0) = \langle D\delta_0, \phi \rangle \end{aligned}$$

Exercice 2. Consider the function:

$$f(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2-t & 1 \leq t \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Compute the first two distributional derivatives of f, i.e. $< Df, - >$ and $< D^2 f, - >$ in $\mathcal{D}'_{\mathbb{R}}$.

Answer: Let ϕ be a compactly supported continuous test function taken from $C_c^\infty(\mathbb{R})$, $f(x)$ defines a distribution on $C_c^\infty(\mathbb{R})$. Hence,

$$\begin{aligned}
\langle Df, \phi \rangle &= -\langle f, \phi' \rangle \\
&= -\int_{-\infty}^{+\infty} f(x)\phi'(x)dx \\
&= -\int_{-\infty}^{0} 0\phi'(x)dx - \int_0^1 x\phi'(x)dx - \int_1^2 (2-x)\phi'(x)dx - \int_2^{+\infty} 0\phi'(x)dx \\
&= -\int_0^1 x\phi'(x)dx - \int_1^2 (2-x)\phi'(x)dx \\
&= \int_0^1 \phi(x)dx - \Big[x\phi(x)\Big]_0^1 + \int_1^2 -\phi(x)dx - \Big[(2-x)\phi(x)\Big]_1^2 \\
&= \int_0^1 \phi(x)dx - \phi(1) + \int_1^2 -\phi(x)dx + \phi(1) \\
&= \int_0^1 \phi(x)dx + \int_1^2 -\phi(x)dx
\end{aligned}$$

therefore

$$Df = \begin{cases} 1 & 0 \leq x \leq 1 \\ -1 & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

the second-order distributional derivative is then computed as

$$\begin{aligned}
\langle D^2 f, \phi \rangle &= \langle D(Df), \phi \rangle \\
&= -\langle Df, \phi' \rangle \\
&= -\int_{-\infty}^{+\infty} D(f(x))\phi'(x)dx \\
&= -\int_0^1 \phi'(x)dx - \int_1^2 (-1)\phi'(x)dx \\
&= -\Big[\phi(x)\Big]_0^1 - \Big[-\phi(x)\Big]_1^2 \\
&= -\phi(1) + \phi(0) - [-\phi(2) + \phi(1)] \\
&= \phi(0) + \phi(2) - 2\phi(1) \\
&= \int_{-\infty}^{+\infty} (\delta(x) + \delta(x-2) - 2\delta(x-1))\phi(x)dx
\end{aligned}$$

hence

$$D^2 f = \delta(x) + \delta(x-2) - 2\delta(x-1)$$

Exercice 3. Let $f \in \mathcal{D}'_{\mathbb{R}}$ be the k-th distributional derivative of some continuous function h with $\text{supp}(h) \subset [0, +\infty[$. Using the Laplace transform, solve the following differential equation in $\mathcal{D}'_{\mathbb{R}}$

$$\langle Du, \cdot \rangle + a\langle u, \cdot \rangle = \langle f, \cdot \rangle$$

HINTS: Recall that one can define the convolution of distributions, and that for distributions as well as for functions, the Laplace transform of the convolution is the product of the Laplace transforms.

Answer: Given the assumption on the distribution f, its Laplace transform $L(f)$ exists. Consequently we have

$$L(f)(s) = \langle f, e^{-st} \rangle = \langle Du, e^{-st} \rangle + a\langle u, e^{-st} \rangle$$

that is

$$L(f)(s) = L(Du)(s) + aL(u)(s).$$

Since $L(Du)(s) = sL(u)(s)$, we get

$$sL(u)(s) + aL(u)(s) = L(f)(s)$$

hence

$$L(u)(s) = \frac{L(f)(s)}{s+a}.$$

Considering that the Laplace transform of a convolution is the product of the Laplace transforms we get

$$u = f * e^{-at}.$$

Exercice 4. Let consider the piece-wise continuous function $G_y(x) = -\frac{1}{2}|x-y|$. Verify that $G_y(x)$ satisfied the following identity in $\mathcal{D}'_{\mathbb{R}}$:

$$\left\langle \frac{\partial^2}{\partial x^2} G, \cdot \right\rangle = -\langle \delta_y, \cdot \rangle$$

where $\langle \delta_y, \cdot \rangle : \mathcal{D} \to \mathbb{R}$ is the Dirac mass distribution concentrated at y,i.e. it is defined by

$$\langle \delta_y, \varphi \rangle = \varphi(y)$$

Answer: The second derivative $\frac{d^2 G_y}{dx^2}$ of the distribution $G_y \in \mathcal{D}'(\mathbb{R})$ is defined by:

$$\langle \frac{d^2 G_y}{dx^2}, \phi \rangle = \langle G_y, \phi'' \rangle$$

for all test functions $\phi \in \mathcal{D}(\mathbb{R})$. By definition, $G_y(x) = \frac{1}{2}(x-y)$ on $]-\infty, y]$ and $G_y(x) = \frac{1}{2}(y-x)$ on $[y, +\infty[$, hence

$$\langle G_y, \phi'' \rangle = \frac{1}{2}\left[\int_{-\infty}^{y} (x-y)\phi''(x)dx + \int_{y}^{+\infty} (y-x)\phi''(x)dx\right].$$

By using integration by part, and by recalling that the test functions $\phi \in \mathcal{D}(\mathbb{R})$ and their derivatives vanish at $\pm\infty$, we get

$$\begin{aligned}\int_{-\infty}^{y} (x-y)\phi''(x)dx &= -\int_{-\infty}^{y} \phi'(x)dx + \left[(x-y)\phi'(x)\right]_{-\infty}^{y} \\ &= -\left[\phi(x)\right]_{-\infty}^{y} + [0-0] \\ &= -\left[\phi(y) - 0\right] \\ &= -\phi(y).\end{aligned}$$

Similarly

$$\begin{aligned}\int_{y}^{+\infty} (y-x)\phi''(x)dx &= \int_{y}^{+\infty} \phi'(x)dx + \left[(y-x)\phi'(x)\right]_{y}^{+\infty} \\ &= \left[\phi(x)\right]_{y}^{+\infty} + [0-0] \\ &= \left[0 - \phi(y)\right] \\ &= -\phi(y).\end{aligned}$$

Finally we get

$$\langle \frac{d^2 G_y}{dx^2}, \phi \rangle = \frac{1}{2}\left[-\phi(y) - \phi(y)\right] = -\phi(y) = -\langle \delta_y, \phi \rangle.$$