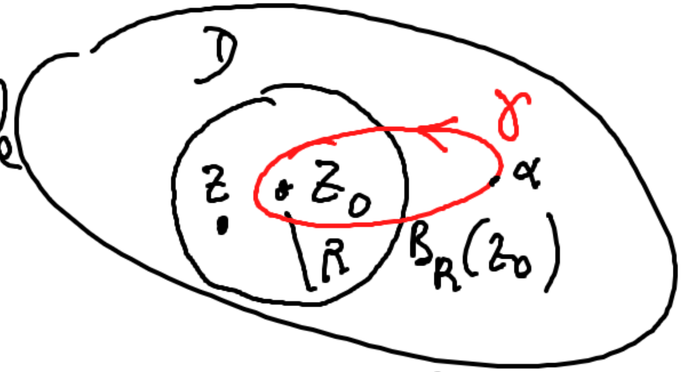
Chap 12 : Thm des résidus, applications

Chap 11 : Série de Laurent

D simplement connexe, $z_0 \in D$, $f : D\setminus\{z_0\} \to \mathbb{C}$ holomorphe



R tq $B_R(z_0) \subset D$ $\quad \forall z \in B_R(z_0)$

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-z_0)^n = \dots + \frac{c_{-n}}{(z-z_0)^n} + \dots + \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \dots + c_n(z-z_0)^n + \dots$$

avec $c_n = \frac{1}{2\pi i}\int_\gamma \frac{f(\alpha)}{(\alpha - z_0)^{n+1}}\,d\alpha \qquad c_{-1} = \frac{1}{2\pi i}\int_\gamma f(\alpha)\,d\alpha = \mathrm{Res}_{z_0}(f)$

z_0 pôle d'ordre m : $f(z) = \sum_{n=-m}^{+\infty} c_n (z-z_0)^n = \frac{c_{-m}}{(z-z_0)^m} + \dots + \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \dots + c_n(z-z_0)^n + \dots$

$$\mathrm{Res}_{z_0}(f) = \lim_{z\to z_0} \frac{d^{m-1}}{dz^{m-1}}(z-z_0)^m f(z) \cdot \frac{1}{(m-1)!}$$

$\left(m = 1 \qquad f(z) = \frac{c_{-1}}{z-z_0} + c_0 + c_1(z-z_0) + \dots \qquad (z-z_0) f(z) = c_{-1} + c_0(z-z_0) \xrightarrow{z\to z_0} c_{-1} \right)$

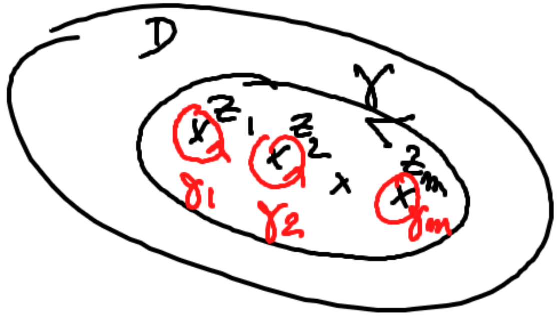
Thm résidu: D simplement connexe

γ courbe fermée $\subset D$

$z_1, z_2, \dots, z_m \subset \operatorname{int} \gamma$

$f : D \setminus \{z_1, z_2, \dots, z_m\}$ holomorphe

$$\int_\gamma f(z)\,dz = 2\pi i \left(\operatorname{Res}_{z_1}(f) + \dots + \operatorname{Res}_{z_m}(f) \right)$$



(Coroll thm Cauchy $\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \dots + \int_{\gamma_m} f(z)\,dz$)

Cas général : $\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$

γ cercle de centre 0 rayon 1 $\quad \gamma(t) = e^{i\theta} d\theta$, $\quad 0 \le \theta \le 2\pi$

$\int_\gamma \tilde{f}(z)\,dz \quad \tilde{f}$?

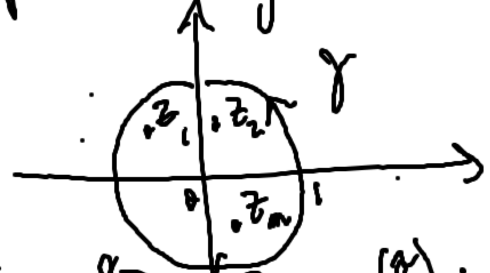
$$\int_\gamma \tilde{f}(z)\,dz = \int_0^{2\pi} \tilde{f}(e^{i\theta})\, ie^{i\theta} d\theta = \int_0^{2\pi} f(\cos\theta, \sin\theta)\,d\theta$$

$$= \int_0^{2\pi} f\left(\frac{e^{i\theta}+e^{-i\theta}}{2}, \frac{e^{i\theta}-e^{-i\theta}}{2i}\right) d\theta$$

on identifie en posant $z = e^{i\theta}$:

$$\tilde{f}(z)\, iz = f\left(\frac{z+\frac{1}{z}}{2}, \frac{z-\frac{1}{z}}{2i}\right)$$

Soit $z_1, z_2, \dots, z_m$ les points singuliers de $\tilde{f}(z)$ contenus dans int γ



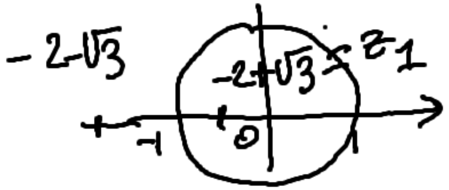
Thm résidus : $\int_\gamma \tilde{f}(z)\,dz = 2\pi i \left(\mathrm{Res}_{z_1}(\tilde{f}) + \dots + \mathrm{Res}_{z_m}(\tilde{f})\right)$

Cas part : $\int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta$

$$\tilde{f}(z)\, iz = \frac{1}{2 + \frac{z+\frac{1}{z}}{2}} \qquad \tilde{f}(z) = \frac{2}{\left(4+z+\frac{1}{z}\right) iz} = \frac{2}{i(z^2+4z+1)}$$

racine de z^2+4z+1 : $\Delta = 16 - 4 = 12$

racine $\frac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$



$\mathrm{Res}_{z_1}(\tilde{f}) \qquad z_1$ pôle d'ordre 1

$$z^2+4z+1 = (z-(-2+\sqrt{3}))(z-(-2-\sqrt{3}))$$

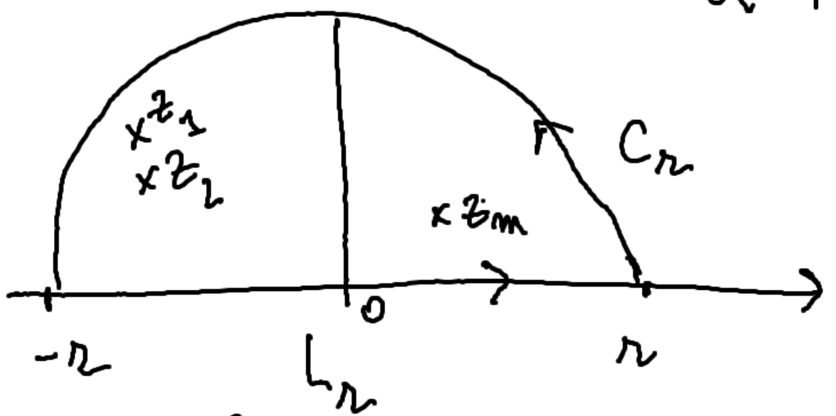
$$(z-z_1)\tilde{f}(z) = \frac{2}{i(z-(-2-\sqrt{3}))} \xrightarrow{z \to z_1} \frac{2}{i(-2+\sqrt{3}-(-2-\sqrt{3}))} = \frac{1}{i\sqrt{3}}$$

Thm résidu : $\int_0^{2\pi} \frac{1}{2+\cos\theta}\, d\theta = \int_\gamma \tilde{f}(z)\,dz = 2\pi i\, \mathrm{Res}_{z_1}(\tilde{f}) = \frac{2\pi i}{i\sqrt{3}} = \frac{2\pi}{\sqrt{3}}$

Cas général $\int_{-\infty}^{+\infty} R(x)\, e^{iax}\, dx \qquad a \geq 0$

et $R(x) = \dfrac{P(x)}{Q(x)}$ P, Q polynômes tels que

- $\deg Q \geq \deg P + 2$
- $Q(x) \neq 0 \quad \forall x \in \mathbb{R}$



(r est destiné à tendre vers l'infini)

$L_r = \{ z \in \mathbb{C} \, ; \, \operatorname{Im} z = 0, \; -r \leq \operatorname{Re} z \leq r \}$

$C_r = \{ z \in \mathbb{C} \, ; \, z = re^{i\theta}, \; 0 \leq \theta \leq \pi \}$

$\gamma_r = L_r \cup C_r$

Thm résidus $\displaystyle\int_{\gamma_r} R(z) e^{iaz}\, dz = 2\pi i \left(\operatorname{Res}_{z_1} \left(R(z) e^{iaz} \right) + \dots + \operatorname{Res}_{z_m} \left(R(z) e^{iaz} \right) \right)$

où $z_1, z_2 \dots z_m$ sont les points singuliers de $R(z) e^{iaz}$

Cas part : $\displaystyle\int_{-\infty}^{+\infty} \frac{x^2}{16 + x^4}\, dx$

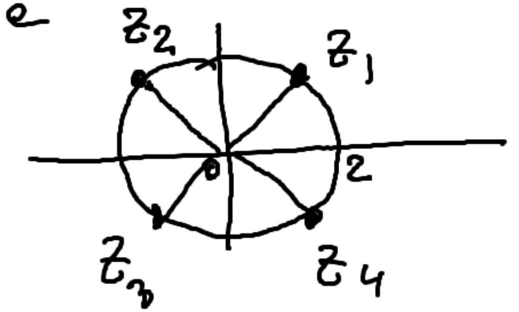
$16 + x^4 \geq 16 \neq 0$

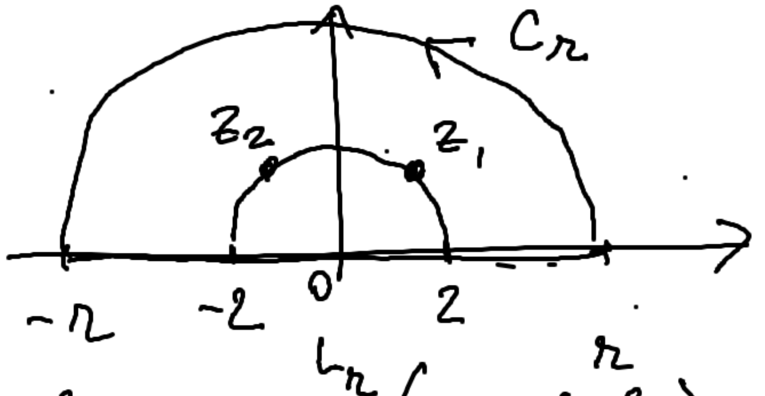
$R(z) e^{iaz} = \dfrac{z^2}{16 + z^4}$ racine de $16 + z^4$:

$z^4 = -16 \quad z = re^{i\theta} \quad r^4 e^{i4\theta} = 2^4 e^{i(\pi + 2k\pi)}$

$r = 2 \quad \theta = \dfrac{\pi}{4} + \dfrac{k\pi}{2}$

$16 + z^4 = (z - z_1)(z - z_2)(z - z_3)(z - z_4)$





$\displaystyle\int_{\gamma_r} \frac{z^2}{16 + z^4}\, dz = 2\pi i \left(\operatorname{Res}_{z_1} \left(\frac{z^2}{16 + z^4} \right) + \operatorname{Res}_{z_2} \left(\frac{z^2}{16 + z^4} \right) \right)$

$$\int_{\gamma_r} R(z)e^{iaz}\,dz = 2\pi i\left(\operatorname{Res}_{z_1}\left(R(z)e^{iaz}\right)+\dots+\operatorname{Res}_{z_m}\left(R(z)e^{iaz}\right)\right)$$

$$\underbrace{\int_{-r}^{r} R(x)e^{iax}\,dx}_{r\to\infty\ \int_{-\infty}^{+\infty} R(x)e^{iax}dx} + \int_{C_r} R(z)e^{iaz}\,dz$$

on va montrer que $\displaystyle\int_{C_r} R(z)e^{iaz}dz \xrightarrow{r\to\infty} 0$

$$\int_{\gamma_r}\frac{z^2}{16+z^4}\,dz = 2\pi i\left(\operatorname{Res}_{z_1}\left(\frac{z^2}{16+z^4}\right)+\operatorname{Res}_{z_2}\left(\frac{z^2}{16+z^4}\right)\right)$$

$$\frac{z^2}{16+z^4} = \frac{z^2}{(z-z_1)(z-z_2)(z-z_3)(z-z_4)}$$

$$(z-z_1)\frac{z^2}{16+z^4} = \frac{z^2}{(z-z_2)(z-z_3)(z-z_4)} \xrightarrow{z\to z_1} \underbrace{\frac{z_1^2}{(z_1-z_2)(z_1-z_3)(z_1-z_4)}}_{\operatorname{Res}_{z_1}\left(\frac{z^2}{16+z^4}\right)}$$

Après calcul on trouve $\dfrac{1+i}{8i\sqrt{2}} = \operatorname{Res}_{z_1}\left(\dfrac{z^2}{16+z^4}\right)$

De même on trouve $\operatorname{Res}_{z_2}\left(\dfrac{z^2}{16+z^4}\right) = \dfrac{1-i}{8i\sqrt{2}}$

$$\underbrace{\int_{\gamma_r}\frac{z^2}{16+z^4}\,dz}_{\int_{-r}^{r}\frac{x^2}{16+x^4}dx + \int_{C_r}\frac{z^2}{16+z^4}dz} = 2\pi i\left(\frac{1+i}{8i\sqrt{2}}+\frac{1-i}{8i\sqrt{2}}\right) = \frac{\pi}{2\sqrt{2}}$$

On va dem que $\displaystyle\int_{C_r}\frac{z^2}{16+z^4}dz \xrightarrow{r\to\infty} 0$

$$\left|\int_{C_r} R(z)e^{iaz}dz\right| = \left|\int_0^{\pi} \frac{P(re^{i\theta})}{Q(re^{i\theta})} e^{iar(\cos\theta + i\sin\theta)} ire^{i\theta} d\theta\right|$$

$\gamma(t) = re^{i\theta}$ $0 \leq \theta \leq \pi$

$$\leq \int_0^{\pi} \frac{|P(re^{i\theta})|}{|Q(re^{i\theta})|} \underbrace{|e^{iar\cos\theta}|}_{1} \underbrace{e^{-ar\sin\theta}}_{\leq 1} r \underbrace{|e^{i\theta}|}_{1} d\theta$$

Puisque $\deg Q \geqslant \deg P + 2$

$$\frac{|P(re^{i\theta})|}{|Q(re^{i\theta})|} \leq \frac{C}{r^2}$$ (pas vraiment démontré) $\to$

donc $$\left|\int_{C_r} R(z)e^{iaz}dz\right| \leq \int_0^{\pi} \frac{rC}{r^2} d\theta = \frac{C}{r}\pi \xrightarrow[r\to\infty]{} 0$$

Conclusion $$\int_{-\infty}^{+\infty} R(x)e^{iax}dx = 2\pi i\left(\mathrm{Res}_{z_1}(R(z)e^{iaz}) + \dots + \mathrm{Res}_{z_m}(R(z)e^{iaz})\right)$$

$$\left|\int_{C_r} \frac{z^2}{16+z^4}dz\right| = \left|\int_0^{\pi} \frac{r^2e^{i2\theta}}{16+r^4e^{i4\theta}} ire^{i\theta} d\theta\right|$$

$$\leq \int_0^{\pi} \frac{r^2}{|16+r^4e^{i4\theta}|} r\, d\theta$$

$\frac{1}{|16+r^4e^{i4\theta}|} \leq$ quelque chose i.e. $|16+r^4e^{i4\theta}| \geqslant$ autre chose

On a : $|r^4e^{i4\theta}| = |r^4e^{i4\theta} + 16 - 16|$
$\leq |r^4e^{i4\theta} + 16| + |16|$

$|16+r^4e^{i4\theta}| \geqslant |r^4e^{i4\theta}| - 16 = r^4 - 16$ (>0 si $r>2$)

$$\frac{1}{|16+r^4e^{i4\theta}|} \leq \frac{1}{r^4-16}$$ (si $r>2$)

$$\left|\int_{C_r} \frac{z^2}{16+z^4}dz\right| \leq \int_0^{\pi} \frac{r^3}{r^4-16} d\theta = \frac{\pi r^3}{r^4-16} \xrightarrow{r\to\infty} 0$$

Conclusion : $\int_{-\infty}^{+\infty} \frac{x^2}{16+x^4}dx = \frac{\pi}{2\sqrt{2}}$

Résumé Analyse complexe

- $f : O \subset \mathbb{C} \to \mathbb{C}$

  $z \to f(z)$

  $x+iy \to u(x,y)+iv(x,y)$

  $u : (x,y) \to u(x,y) \in \mathbb{R}$

  $v \quad\quad\quad\quad v \in \mathbb{R}$

  f holomorphe dans O : $\forall z_0 \in O$ $\lim\limits_{z \to z_0} \dfrac{f(z)-f(z_0)}{z-z_0}$ existe la même $\forall z \to z_0$ $\iff$ $u, v \in C^1$ $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ et $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

  $$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) - i\frac{\partial u}{\partial y}(x,y)$$

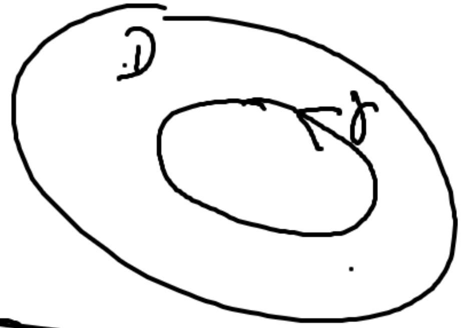
- Thm Cauchy

  D simplement connexe

  $f : D \to \mathbb{C}$ holomorphe

  γ courbe simple fermée régulière (par morceaux) $\subset D$

  $\int_\gamma f(z)\,dz = 0$



- Corollaire

  D simplement connexe

  γ courbe fermée. $z_0 \in \operatorname{int} \gamma$

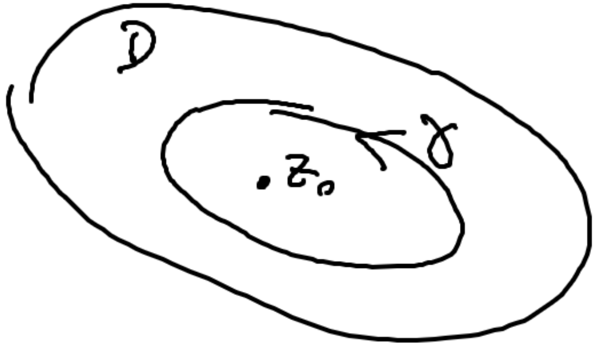
  $f : D \setminus \{z_0\} \to \mathbb{C}$ holomorphe

  $$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$

on applique le thm de Cauchy dans la zone hachurée en bleue, puis celle en vert et on somme.

$f : D \setminus \{z_1, z_2, \dots, z_m\}$ holomorphe

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \dots + \int_{\gamma_m} f(z)\,dz$$



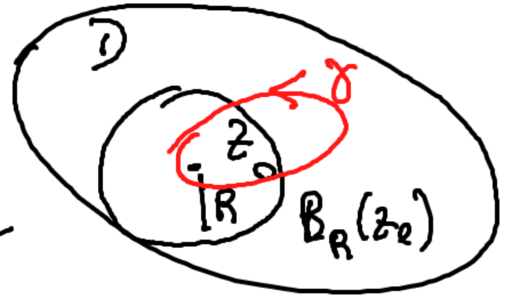
• Formules intégrales de Cauchy : D simplement connexe
$f : D \to \mathbb{C}$ holomorphe
γ courbe fermée $\subset D$
$z_0 \in \operatorname{int} \gamma$

$$\int_\gamma \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$$ (Corollaire thm Cauchy)

$$\int_\gamma \frac{f(z)}{(z-z_0)^2}\,dz = 2\pi i\, f'(z_0) \qquad f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

$$n! \int_\gamma \frac{f(z)}{(z-z_0)^{n+1}}\,dz = 2\pi i\, f^{(n)}(z_0)$$

- Série de Taylor



D simplement connexe $f: D \to \mathbb{C}$ holomorphe

$z_0 \in D$

R tq $B_R(z_0) \subset D$

$z \in B_R(z_0)$

$$f(z) = \sum_{n=0}^{+\infty} \underbrace{\frac{f^{(n)}(z_0)}{n!}}_{} (z-z_0)^n$$

(dans $\mathbb{R}$: $e^x = 1 + x + \frac{x^2}{2} + \dots + \frac{x^n}{n!} + \dots = \sum_{n=0}^{+\infty} \frac{x^n}{n!}$)

$$c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\alpha)}{(\alpha - z_0)^{n+1}} \, d\alpha$$ où γ courbe fermée tq $z_0 \in \operatorname{int} \gamma$

- Série de Laurent $f : D \setminus \{z_0\} \to \mathbb{C}$ holomorphe (singulière en z_0)

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z - z_0)^n \qquad c_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\alpha)}{(\alpha - z_0)^{n+1}} \, d\alpha$$