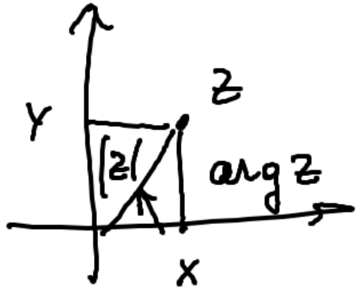
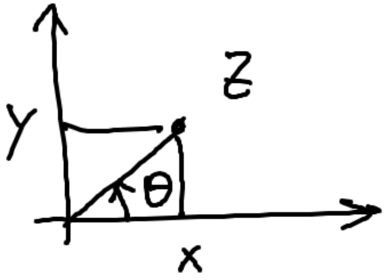
Chap 9 : fonctions holomorphes - Eq. Cauchy-Riemann



Rappels : i tq $i^2=-1$ $\quad e^{i\theta}=\cos\theta+i\sin\theta \quad \theta\in\mathbb{R}$

- $z\in\mathbb{C}$ $\quad z=x+iy \quad \mathrm{Re}\,z=x \quad \mathrm{Im}\,z=y$
- — $\quad \bar{z}=x-iy$ conjugué complexe
- — $\quad |z|=\sqrt{x^2+y^2}$ module argument $\arg z$

$z=|z|e^{i\arg z}$ $\quad \arg z$ déf à $2k\pi$ près $k\in\mathbb{Z}$



valeur principale de l'argument $\quad -\pi<\arg z\leq\pi$

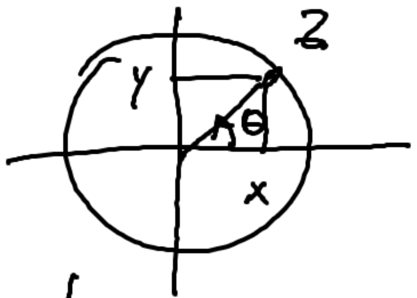
Ex: $z=z_1+z_2=x_1+iy_1+x_2+iy_2=x_1+x_2+i(y_1+y_2)$

$z_1z_2=(x_1+iy_1)(x_2+iy_2)=x_1x_2-y_1y_2+i(x_1y_2+x_2y_1)$

$z\bar{z}=(x+iy)(x-iy)=x^2+y^2=|z|^2$

$z = |z| e^{i\theta}$ $\qquad x = |z|\cos\theta$

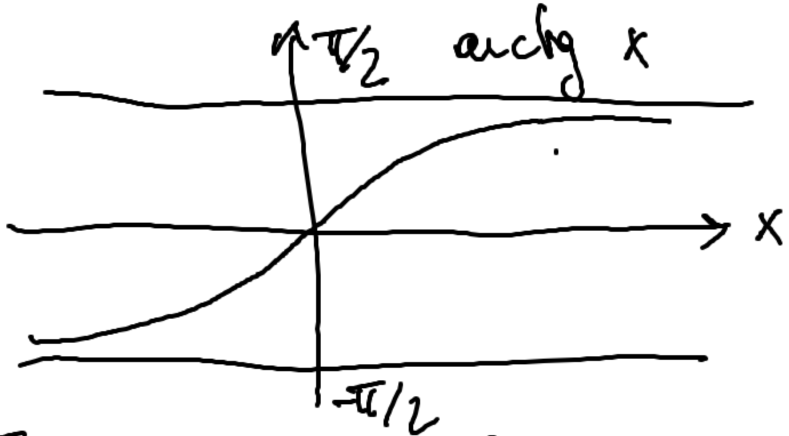
$\quad = x + iy$ $\qquad y = |z|\sin\theta$

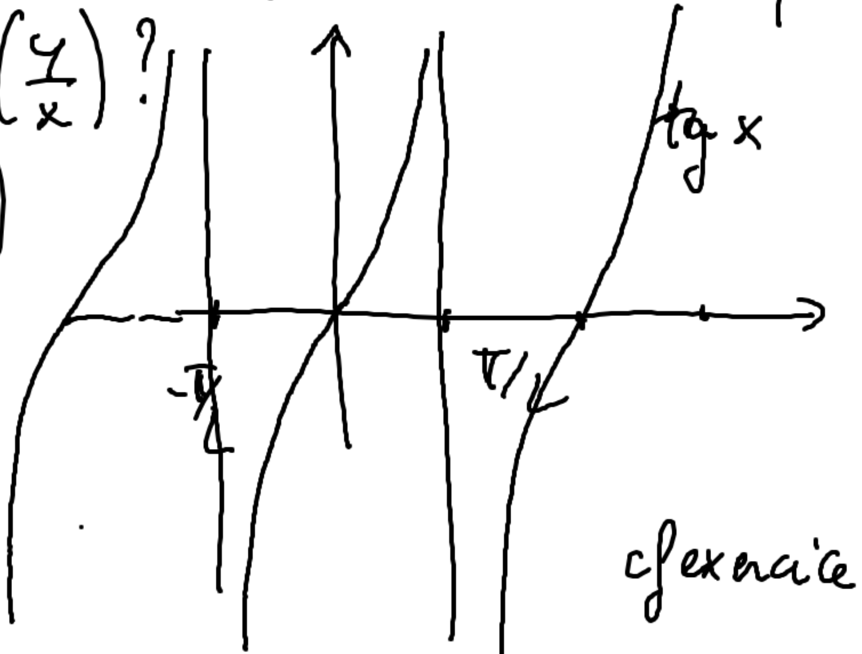


Peut-on exprimer θ en fonction de x y ?

$\operatorname{tg}\theta = \frac{y}{x}$ $\qquad \theta = \operatorname{arctg}\left(\frac{y}{x}\right)$?

Si $x > 0$ $\qquad \theta = \operatorname{arctg}\left(\frac{y}{x}\right)$

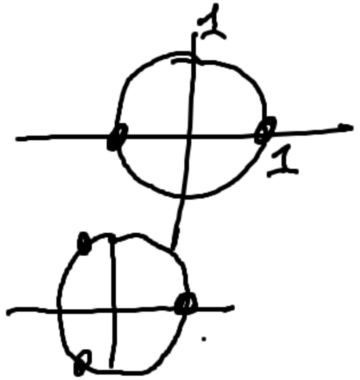




cf exercice

Ex: $z \in \mathbb{C}$ tq $z^2 = 1$ $\qquad z = |z| e^{i\theta}$ $\qquad z^2 = |z|^2 e^{i2\theta} = 1$

$|z|^2 = 1$ $\qquad e^{i2\theta} = 1$ $\qquad 2\theta = 2k\pi$ $\quad k \in \mathbb{Z}$



$z \in \mathbb{C}$ tq $z^3 = 1$ $\qquad z^3 = |z|^3 e^{i3\theta} = 1$

$|z|^2 = 1$ $\qquad 3\theta = 2k\pi$ $\quad k \in \mathbb{Z}$

fonctions complexes

$$f : \mathbb{C} \to \mathbb{C}$$
$$z \to f(z)$$

$u : \mathbb{R}^2 \to \mathbb{R}$ $\quad$ $v : \mathbb{R}^2 \to \mathbb{R}$

$(x,y) \to u(x,y)$ $\quad$ $(x,y) \to v(x,y)$

$$z = x+iy \to u(x,y) + iv(x,y)$$

Ex1: $f(z) = \bar{z} = x - iy$ $\quad u(x,y) = x \quad v(x,y) = -y$

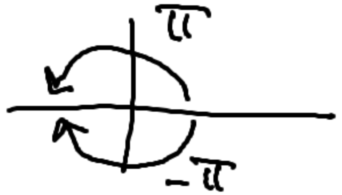
Ex2: $f(z) = z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{u(x,y)} + i \underbrace{2xy}_{v(x,y)}$

Ex3: $f(z) = \dfrac{1}{z} = \dfrac{1}{x+iy} = \dfrac{x-iy}{x^2+y^2} = \underbrace{\dfrac{x}{x^2+y^2}}_{u(x,y)} + i \left(\underbrace{-\dfrac{y}{x^2+y^2}}_{v(x,y)} \right)$

Ex4: $f(z) = e^z = e^{x+iy} = e^x e^{iy} = \underbrace{e^x \cos y}_{u(x,y)} + i \underbrace{e^x \sin y}_{v(x,y)}$

<u>Ex 5</u> : logarithme complexe : "fonction inverse de e^z"

$z = |z| e^{i\arg z}$ $\qquad -\pi < \arg z \le \pi$



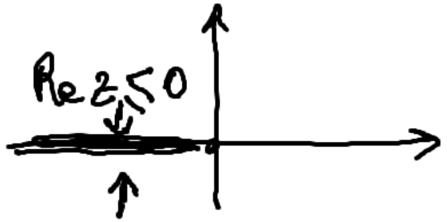
$\left(\log(ab) = \log a + \log b \quad a,b > 0 \quad \log e = 1 \right)$

$\log z = \log|z| + i\arg z$

On a bien $e^{\log z} = e^{\log|z| + i\arg z} = e^{\log|z|} e^{i\arg z} = |z| e^{i\arg z} = z$

attention : • $\log : \mathbb{C} \setminus \{0\} \to \mathbb{C}$

• $\log(z_1 z_2)$ peut être diff de $\log z_1 + \log z_2$



ex : $z_1 = -1 \quad z_2 = -1$

$\log z_1 z_2 = \log 1 = 0 \quad \log z_1 = \log z_2 = i\pi$

• log n'est pas continue sur $\mathbb{C} \setminus \{0\}$

Ex6: $\gamma \in \mathbb{R}$ $\quad f(z) = z^{\gamma} = e^{\gamma \log z} = e^{\gamma(\log|z| + i\arg z)} = \underbrace{e^{\gamma \log|z|}}_{|z|^{\gamma}} e^{\gamma i \arg z}$

Ex7: $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$ $\quad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$\cosh z = \dfrac{e^{z} + e^{-z}}{2}$ $\quad \sinh z = \dfrac{e^{z} - e^{-z}}{2}$

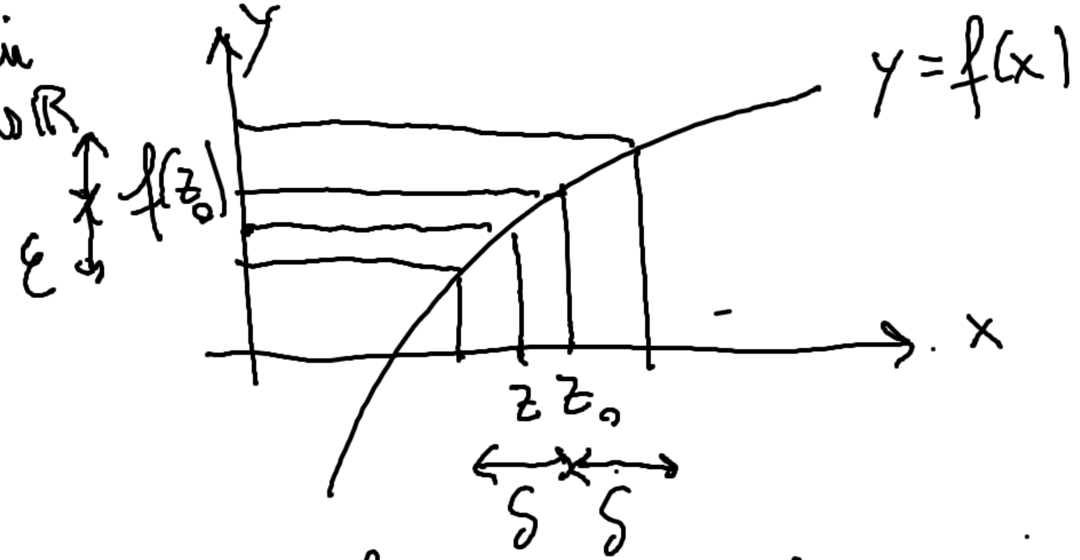
$\cos z = \dfrac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} = \dfrac{e^{-y} e^{ix} + e^{y} e^{-ix}}{2} = \dfrac{e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x)}{2}$

$= \underbrace{\cos x \cosh y}_{u(x,y)} + i\underbrace{(\sin x \sinh y)}_{v(x,y)}$.

Limites : Déf : $f : \mathbb{C} \to \mathbb{C}$ est continue en z_0 et on écrit $\lim\limits_{z \to z_0} f(z) = f(z_0)$ si

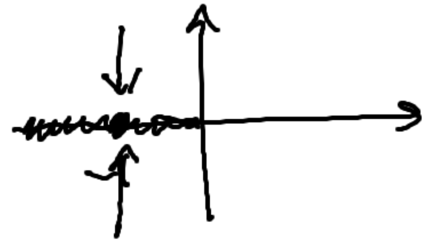
$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \text{si } |z - z_0| < \delta \text{ alors } |f(z) - f(z_0)| < \varepsilon$$

dessin dans $\mathbb{R}$



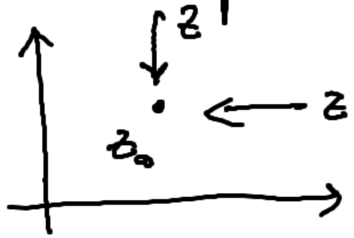
Si f et g sont continues en z_0 alors $f+g$, $f \cdot g$ est cont en z_0

Ex $\log z = \log|z| + i \arg z$ n'est pas cont en $z_0 = -1$



Dérivées : Def 9.1 livre : $f : \mathbb{C} \to \mathbb{C}$ dérivable en z_0 si

$$\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$ existe et est finie (la même limite pour tout $z \to z_0$)



Dans ce cas on note $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$

Si f est dérivable $\forall z_0 \in \mathbb{C}$ on dit que f est holomorphe

Idem $f : O \to \mathbb{C}$ $\quad O$ ouvert $\subset \mathbb{C}$

On veut montrer par exemple que $(e^z)' = e^z$.

$$\frac{e^z - e^{z_0}}{z-z_0} = \frac{e^{x+iy} - e^{x_0+iy_0}}{x-x_0+i(y-y_0)} = \frac{e^x \cos y - e^{x_0}\cos y_0 + i\left(e^x \sin y - e^{x_0}\sin y_0\right)}{x-x_0+i(y-y_0)} \ ???$$

Thm 9.2 : $O \subset \mathbb{C}$ $\quad f : O \to \mathbb{C}$

$$z \to f(z)$$

$$x+iy \to u(x,y) + i v(x,y)$$

$$\left(f \text{ holomorphe} \right) \iff \left(u,v \in \mathcal{C}^1(O) \text{ et } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ et } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \right)$$

de plus $f'(z) = \frac{\partial u}{\partial x}(x,y) + i \frac{\partial v}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y) - i \frac{\partial u}{\partial y}(x,y)$

Ex 2 : $f(z) = z^2 = x^2 - y^2 + i\, 2xy \qquad \frac{\partial u}{\partial x}(x,y) = 2x = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y}(x,y) = -2y = -\frac{\partial v}{\partial x}(x,y)$

$f'(z) = 2x + i\, 2y = 2(x+iy) = 2z \qquad f : \mathbb{C} \to \mathbb{C}$ holomorphe

Ex 1 : $f(z) = \bar{z} = x - iy \qquad \frac{\partial u}{\partial x} = 1 \qquad \frac{\partial v}{\partial y} = -1$ $\quad f$ n'est pas holomorphe

$$\overbrace{\mathbb{C} \setminus \{0\}}^{O}$$

Ex 3 : $f(z) = \frac{1}{z} \qquad f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$

$= \frac{x}{x^2+y^2} + i \frac{-y}{x^2+y^2} \qquad \frac{\partial u}{\partial x} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} \qquad \frac{\partial v}{\partial y}(x,y) = \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} \qquad \frac{\partial u}{\partial x}(x,y) = \frac{\partial v}{\partial y}(x,y)$

on a aussi

On a aussi $\dfrac{\partial u}{\partial y} = \dfrac{\partial}{\partial y} \dfrac{x}{x^2+y^2} = \dfrac{-x(2y)}{(x^2+y^2)^2}$ $\qquad \dfrac{\partial u}{\partial y}(x,y) = -\dfrac{\partial v}{\partial x}(x,y)$

$$\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{y}{x^2+y^2}\right) = \frac{+2xy}{(x^2+y^2)^2}$$

les 2 cond de Cauchy-Riemann sont satisfaites et on a

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y)$$

$$= \frac{y^2-x^2}{(x^2+y^2)^2} + i\,\frac{2xy}{(x^2+y^2)^2}$$

D'autre part $\dfrac{1}{z^2} = \dfrac{1}{(x+iy)^2} = \dfrac{1}{x^2-y^2+2ixy} = \dfrac{x^2-y^2-2ixy}{(x^2-y^2)^2+4x^2y^2}$

$$= \frac{x^2-y^2-2ixy}{x^4+y^4+2x^2y^2} = \frac{x^2-y^2-2ixy}{(x^2+y^2)^2}$$

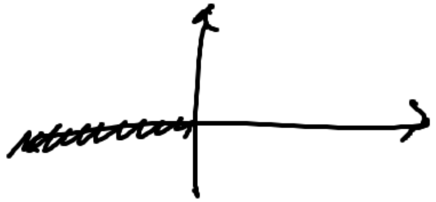
Donc $f'(z) = -\dfrac{1}{z^2}$

Ex 4: $f(z) = e^{z} = \underbrace{e^{x}\cos y}_{u(x,y)} + i\,\underbrace{e^{x}\sin y}_{v(x,y)}$

$$\left.\begin{array}{lll} \dfrac{\partial u}{\partial x}(x,y) = e^{x}\cos y & \dfrac{\partial v}{\partial y}(x,y) = e^{x}\cos y & \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y}(x,y) = -e^{x}\sin y & \dfrac{\partial v}{\partial x}(x,y) = e^{x}\sin y & \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{array}\right\} \begin{array}{l} f \text{ holomorphe} \\ f : \mathbb{C} \to \mathbb{C} \end{array}$$

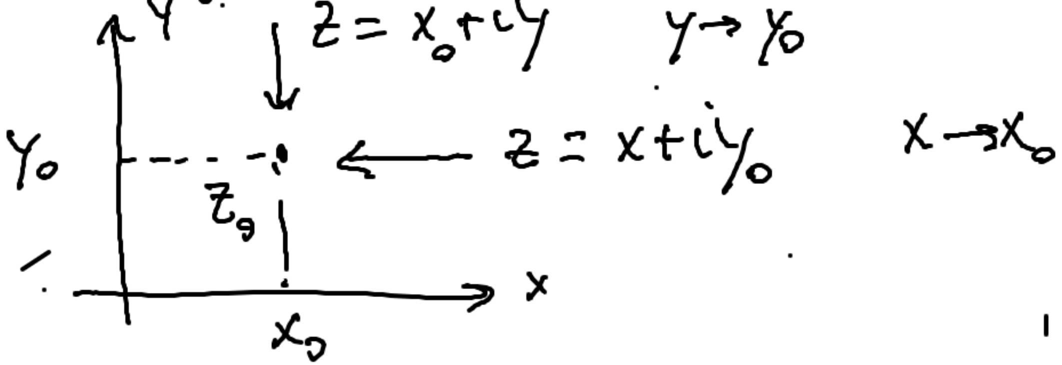
$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\,\frac{\partial v}{\partial x}(x,y) = e^{x}\cos y + i\,e^{x}\sin y = e^{z}$$

Ex 5: $f(z) = \log z$ $\quad$ f holomorphe $O = \mathbb{C} \setminus \left\{ z ;\ \begin{array}{l} \operatorname{Re} z \leq 0 \\ \operatorname{Im} z = 0 \end{array} \right\}$



$$f'(z) = \frac{1}{z} \quad \forall z \in O$$

Ex 6: idem $f(z) = z^{\gamma}$ $\quad$ f holomorphe $O = \mathbb{C} \setminus \{ z ;\ \operatorname{Re} z \leq 0,\ \operatorname{Im} z = 0 \}$



Dem Thm 9.2 quiz

Les règles de dérivation sont les mêmes que dans $\mathbb{R}$

$\mathcal{O} \subset \mathbb{C}$ $\quad f,g : \mathcal{O} \to \mathbb{C}$ holomorphes $\quad \forall z \in \mathcal{O}$ on a :

$$(f+g)'(z_0) = f'(z_0) + g'(z_0)$$

$$(fg)'(z_0) = f'(z_0)\,g(z_0) + f(z_0)\,g'(z_0)$$

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)\,g(z_0) - f(z_0)\,g'(z_0)}{(g(z_0))^2}$$

$$(f\circ g)'(z_0) = f'(g(z_0))\,g'(z_0)$$

Par ex : $\quad \dfrac{f(z)g(z) - f(z_0)g(z_0)}{z-z_0} = \dfrac{\big(f(z)-f(z_0)\big)g(z) + f(z_0)\big(g(z)-g(z_0)\big)}{z-z_0}$

$$(fg)'(z_0) = \lim_{z\to z_0} \underline{\qquad\qquad}$$

$$= \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \lim_{z\to z_0} g(z) + f(z_0) \lim_{z\to z_0} \frac{g(z)-g(z_0)}{z-z_0}$$

$$= f'(z_0)\,g(z_0) + f(z_0)\,g'(z_0)$$