Evaluation : tablette, une seule page.

examen, 2h15 — 10 à 15 questions QCM, 1 ou 2 exercices à rédiger

poids — 2/3 — 1/3

ou — 1/2 — 1/2

transf. de laplace + thm des résidus

formules à savoir (pas les tables des transformées)

Séries de Fourier : chaleur, ondes — 0 — 1 — x — ou $(0, L)$ — $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f$

Transformées de Fourier : — $x \in \mathbb{R}$

L'éq. des ondes et transf de Fourier (pas dans le livre)

Étant donné $f(x,t)$ $x\in\mathbb{R}$, $t>0$, $u_0(x)$, $v_0(x)$ on cherche $u(x,t)$ tq

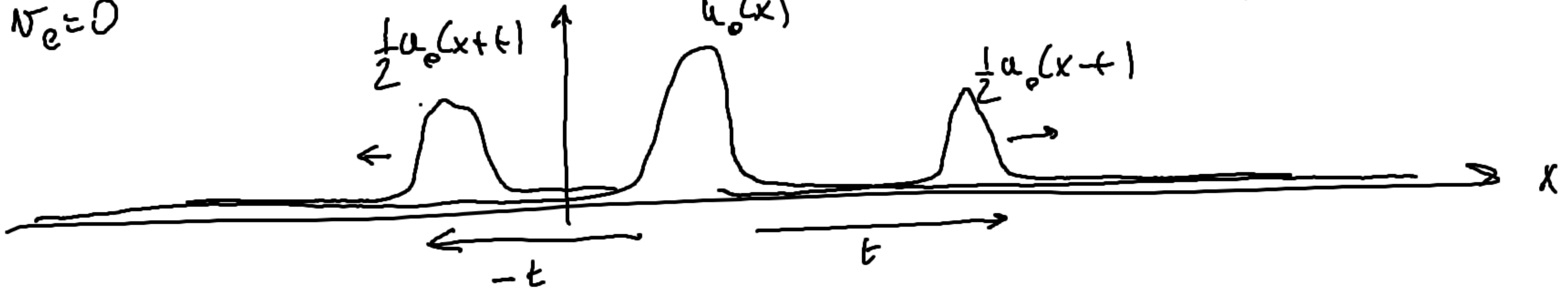
$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = f(x,t) & x\in\mathbb{R} \quad t>0 \qquad \text{éq. ondes} \\ u(x,0) = u_0(x) \\ \frac{\partial u}{\partial t}(x,0) = v_0(x) \end{cases}$$

Remarques : • Pas de conditions aux limites mais $\int_{-\infty}^{+\infty} |u_0(x)|\,dx < +\infty$ et $\int_{-\infty}^{+\infty} |v_0(x)|\,dx < +\infty$

• Si $f(x,t)=0$ la sol est donnée par

$$u(x,t) = \frac{1}{2}\Big(u_0(x-t) + u_0(x+t)\Big) + \frac{1}{2}\int_{x-t}^{x+t} v_0(y)\,dy \qquad \text{Duhamel}$$

$v_0 = 0$



On pose $\hat{u}(\alpha,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x,t)\, e^{-i\alpha x}\, dx \qquad \alpha \in \mathbb{R} \qquad t$ paramètre.

idem pour $\hat{f}(\alpha,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x,t)\, e^{-i\alpha x} dx$ et $\hat{u}_0(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_0(x)\, e^{-i\alpha x} dx$, $\hat{v}_0(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} v_0(x)\, e^{-i\alpha x} dx$

$\hat{u}(\alpha,t)$ inconnu, $\hat{f}(\alpha,t)$, $\hat{u}_0(\alpha)$, $\hat{v}_0(\alpha)$ connus

On multiplie l'eq des ondes par $e^{-i\alpha x}$ $\alpha \in \mathbb{R}$ et on intègre entre $x = -\infty$ et $x = +\infty$:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) \right) e^{-i\alpha x}\, dx = \underbrace{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x,t)\, e^{-i\alpha x}\, dx}_{\hat{f}(\alpha,t)}$$

$$\frac{\partial^2}{\partial t^2} \hat{u}(\alpha,t) - (i\alpha)^2\, \hat{u}(\alpha,t) = \hat{f}(\alpha,t)$$

$f : \mathbb{R} \to \mathbb{R}$

$x \to f(x)$

$\mathcal{F}(f')(\alpha) = i\alpha\, \mathcal{F}(f)(\alpha)$

$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f'(x)\, e^{-i\alpha x} dx$

$= i\alpha \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-i\alpha x} dx$

Condition initiale $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u(x,0)\, e^{-i\alpha x} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} u_0(x)\, e^{-i\alpha x} dx$: $\hat{u}(\alpha,0) = \hat{u}_0(\alpha)$

de même on obtient $\hat{v}(\alpha,0) = \hat{v}_0(\alpha)$

On obtient donc, pour chaque $\alpha \in \mathbb{R}$, l'eq. différentielle :

$$\begin{cases} \frac{\partial^2}{\partial t^2} \hat{u}(\alpha,t) + \alpha^2 \hat{u}(\alpha,t) = \hat{f}(\alpha,t) & t>0 \\ \hat{u}(\alpha,0) = \hat{u}_0(\alpha) \\ \frac{\partial}{\partial t}\hat{u}(\alpha,0) = \hat{v}_0(\alpha) \end{cases}$$

α est un paramètre.

$$\begin{cases} y''(t) + \alpha^2 y(t) = 0 \\ y(0) = 1 \\ y'(0) = 0 \end{cases}$$

$y(t) = \cos(\alpha t)$

On résout cette eq. diff en utilisant la tr. de Laplace :

$$\hat{u}(\alpha,t) = \hat{u}_0(\alpha)\cos(\alpha t) + \frac{\hat{v}_0(\alpha)}{\alpha}\sin(\alpha t) + \int_0^t \hat{f}(\alpha,s)\sin(\alpha(t-s))\,ds$$

On a la formule suivante : pour $a \in \mathbb{R}$

$$\mathcal{F}(f(x+a))(\alpha) = e^{i\alpha a}\mathcal{F}(f)(\alpha)$$

$$\mathcal{F}(f)(\alpha)\cdot\mathcal{F}(g)(\alpha) = \frac{1}{\sqrt{2\pi}}\mathcal{F}(f*g)(\alpha)$$

En effet : $\mathcal{F}(f(x+a))(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x+a)\,e^{-i\alpha x}\,dx$. $\quad x+a=y \quad dx=dy \quad x=y-a$

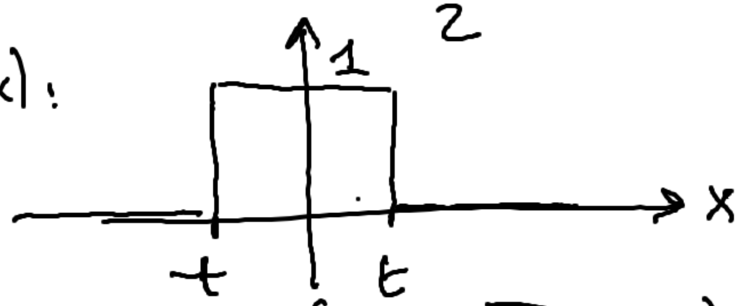
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(y)\,e^{-i\alpha(y-a)}\,dy = e^{i\alpha a}\mathcal{F}(f)(\alpha)$$

On a : $\hat{u}_0(\alpha)\cos(\alpha t) = \hat{u}_0(\alpha)\,\frac{e^{i\alpha t}+e^{-i\alpha t}}{2} = \mathcal{F}(u_0)(\alpha)\,\frac{e^{i\alpha t}+e^{-i\alpha t}}{2}$

$$= \frac{\mathcal{F}(u_0(x+t))(\alpha)+\mathcal{F}(u_0(x-t))(\alpha)}{2}$$

Transformée de Fourier inverse de $\hat{u}_0(\alpha)\cos\alpha t$: $\frac{u_0(x+t)+u_0(x-t)}{2}$

On a : $\mathcal{F}(g_t(x))(\alpha) = \sqrt{\frac{2}{\pi}}\,\frac{\sin(\alpha t)}{\alpha}$ où $g_t(x)$:



cf tables

Donc $\hat{v}_0(\alpha)\,\frac{\sin(\alpha t)}{\alpha} = \mathcal{F}(v_0)(\alpha)\cdot\mathcal{F}\left(\sqrt{\frac{\pi}{2}}\,g_t(x)\right) = \frac{1}{\sqrt{2\pi}}\,\mathcal{F}\left(v_0 * \sqrt{\frac{\pi}{2}}\,g_t(x)\right)$

avec $\left(v_0 * \sqrt{\frac{\pi}{2}}\,g_t\right)(x) = \int_{-\infty}^{+\infty} v_0(y)\sqrt{\frac{\pi}{2}}\,\underbrace{g_t(x-y)}_{1 \text{ si } -t\leq x-y\leq t \text{ ie } x-t\leq y\leq x+t}\,dy$ (déf de $*$: $(f*g)(x) = \int_{-\infty}^{+\infty} f(y)\,g(x-y)\,dy = \int_{-\infty}^{+\infty} f(x-y)\,g(y)\,dy$)

$= \sqrt{\frac{\pi}{2}}\int_{x-t}^{x+t} v_0(y)\,dy$

$\hat{v}_0(\alpha)\,\frac{\sin(\alpha t)}{\alpha} = \mathcal{F}\left(\frac{1}{2}\int_{x-t}^{x+t} v_0(y)\,dy\right)$

$$\hat{u}(\alpha,t) = \hat{u}_0(\alpha)\cos(\alpha t) + \hat{v}_0(\alpha)\frac{\sin(\alpha t)}{\alpha} + \int_0^t \hat{f}(\alpha,s)\sin(\alpha(t-s))\,ds$$

$$= \mathcal{F}\left(\frac{u_0(x-t)+u_0(x+t)}{2}\right) + \mathcal{F}\left(\frac{1}{2}\int_{x-t}^{x+t} v_0(y)\,dy\right) + \int_0^t \hat{f}(\alpha,s)\sin(\alpha(t-s))\,ds$$

Transformée de Fourier inverse (formule d'inversion) $\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\int_0^t \hat{f}(\alpha,s)\sin(\alpha(t-s))\,ds\; e^{i\alpha x}\,d\alpha}$

Solution $u(x,t) = \frac{1}{2}\left(u_0(x-t)+u_0(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} v_0(y)\,dy +$