Exercise 1. Let $H(t)$ be the piece-wise continuous function

$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

and define $u(t) = \cos(t)H(t)$. Verify that $u(t)$ is a solution of the following differential equation in $\mathcal{D}'_{\mathbb{R}}$ (i.e. distributional differential equation):

$$\langle D^2 u, \cdot \rangle + \langle u, \cdot \rangle = \langle D\delta_0, \cdot \rangle$$

where $\langle \delta_0, \cdot \rangle : \mathcal{D} \to \mathbb{R}$ is the Dirac mass distribution, defined by

$$\langle \delta_0, \varphi \rangle = \varphi(0)$$

HINTS: use the definition of distributional derivative; use integration by part; recall that the test functions φ and their derivatives are zero outside a bounded set.

Exercise 2. Consider the function:

$$f(t) = \begin{cases} t & 0 \leq t \leq 1 \\ 2 - t & 1 \leq t \leq \\ 0 & \text{otherwise} \end{cases}$$

Compute the first two distributional derivatives of f, i.e. $< Df, - >$ and $< D^2 f, - >$ in $\mathcal{D}'_{\mathbb{R}}$.

Exercise 3. Using the Laplace transform, solve the following differential equation in $\mathcal{D}'_{\mathbb{R}}$

$$\langle Du, \cdot \rangle + a\langle u, \cdot \rangle = \langle f, \cdot \rangle$$

for some fixed distribution $f \in \mathcal{D}'_{\mathbb{R}}$.

If f is a function, then the equation makes sense also in the usual sense. Write down more explicitly the solution in the classical setting with initial condition $u(0) = u_0$.

HINTS: Recall that one can define convolution of distributions, and that for distributions as well as for functions, the Laplace transform of the convolution is the product of the Laplace transforms.

Exercise 4. Let consider the piece-wise continuous function $G_y(x) = -\frac{1}{2}|x - y|$. Verify that $G_y(x)$ satisfied the following identity in $\mathcal{D}'_{\mathbb{R}}$:

$$\langle \frac{\partial^2}{\partial x^2} G, \cdot \rangle = -\langle \delta_y, \cdot \rangle$$

where $\langle \delta_y, \cdot \rangle : \mathcal{D} \to \mathbb{R}$ is the Dirac mass distribution concentrated at y,i.e. it is defined by

$$\langle \delta_y, \varphi \rangle = \varphi(y)$$