MATH-207(a) Analysis IV
Additional exercises for exam

Exercice 1. See exercise sheets.

Answer. See solutions sheets. ■

Exercice 2. Suppose $y : [0, \infty) \to \mathbb{R}$ satisfies

$$y'''(t) + y(t) = e^{-t}, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -1$$

Find the Laplace transform $\mathcal{L}(y)$.

Answer. Let us denote $Y(s) = \mathcal{L}(y)(s)$. Applying the Laplace transform to the equation and using the initial conditions we find

$$\begin{aligned} s^3Y(s) - s^2y(0) - sy'(0) - y''(0) + Y(s) &= \frac{1}{s+1}, \\ \Leftrightarrow \quad (s^3+1)Y(s) - s^2 + 1 &= \frac{1}{s+1}, \\ \Leftrightarrow \quad Y(s) &= \frac{(s^2-1)(s+1)+1}{(s+1)^3(s+1)}. \end{aligned}$$

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Exercice 3. Given the curve

$$\gamma : [0, 2\pi] \to \mathbb{C}, \quad \theta \mapsto 2e^{\theta \mathrm{i}}$$

compute the curve integrals

$$\int_\gamma \frac{(z+1)^3}{z} \, \mathrm{d}z, \quad \int_\gamma \frac{\sin(z)}{z} \, \mathrm{d}z, \quad \int_\gamma \frac{\cosh(z)}{z} \, \mathrm{d}z.$$

Answer. For the three curves we can use the residue theorem. All the integrands have only one pole in $z_0 = 0$ that is enclosed by the closed curve (a circle of radius 2 centered in the origin).

Therefore, it is sufficient to compute the residue at the origin for each of the integrands to compute the integrals. We find

$$\begin{aligned}
\mathrm{Res}_0\left(\frac{(z+1)^3}{z}\right) &= \lim_{z\to 0} z\frac{(z+1)^3}{z} = 1. \quad \text{(pole or order 1)}\\
\mathrm{Res}_0\left(\frac{\sin(z)}{z}\right) &= 0 \qquad \text{(removable singularity)}\\
\mathrm{Res}_0\left(\frac{\cosh(z)}{z}\right) &= \lim_{z\to 0} z\frac{\cosh(z)}{z} = 1. \quad \text{(pole or order 1)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_\gamma \frac{(z+1)^3}{z}\,\mathrm{d}z &= 2\pi i\\
\int_\gamma \frac{\sin(z)}{z}\,\mathrm{d}z &= 0\\
\int_\gamma \frac{\cosh(z)}{z}\,\mathrm{d}z &= 2\pi i
\end{aligned}$$

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Exercice 4. Find the preimage under the Laplace transform of the function

$$F(z) = \frac{3}{(z-1)^2} + \frac{5}{(z-3)^2-1}.$$

Answer. Exploiting the table of known Laplace transforms and its properties we find that

$$f(t) := \mathcal{L}^{-1}(F)(t) = 3te^t + 5e^{3t}\sinh(t).$$

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Exercice 5. Compute the Laplace transform of the following functions:

$$\begin{aligned}
f(t) &= t^3 + \sin(10t) + e^{6t}\cos(5t) + \int_0^t e^{-t}\cosh(t)\mathrm{d}t\\
g(t) &= \int_0^t \int_0^u e^{-2v}\sinh(v)\mathrm{d}v\ \mathrm{d}u\\
h(t) &= \int_0^t \sin(s)\cos(t-s)\mathrm{d}s,\\
j(t) &= e^{-2t} - 2te^{-2t}
\end{aligned}$$

The functions are understood to be zero for non-positive $t < 0$.

Answer. Exploiting the table of known Laplace transforms and its properties we find that

$$\begin{aligned}
F(s) &= \frac{6}{s^4} + \frac{10}{s^2+100} + \frac{s-6}{(s-6)^2+25} + \frac{s+1}{s^2(s+2)},\\
G(s) &= \frac{1}{s^2(s+3)(s+1)},\\
H(s) &= \frac{s}{(s^2+1)^2},\\
J(s) &= \frac{1}{s+2} - \frac{2}{(s+2)^2}.
\end{aligned}$$

For F we exploited the linearity of Laplace transform and for the last term the property of the Laplace transform of an integral. For G we exploited two times sequentially the property of the Laplace transform of an integral: do the Laplace transform of the inner integral then of the outer integral. For H it is the Laplace transform of a convolution. For J there is nothing special going on.

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