Exam training I

Exercice 1. Find a holomorphic function $f : \mathbb{C} \to \mathbb{C}$ such that its real part is:

$$u(x, y) = x^2 - y^2 + e^{-x} \cos y$$

Answer: Using the Cauchy-Riemann equations we get

$$\begin{cases} u_x = 2x - e^{-x} \cos y = v_y \\ u_y = -2y - e^{-x} \sin y = -v_x \end{cases} \Leftrightarrow \begin{cases} v_y = 2x - e^{-x} \cos y \\ v_x = 2y + e^{-x} \sin y \end{cases}$$

From the second equation we get

$$v(x, y) = \alpha(y) + 2xy - e^{-x} \sin y.$$

By taking the derivative of v with respect to y, and using the first equation, we get

$$\alpha'(y) = 0 \quad \Rightarrow \quad \alpha(y) = \alpha_0 = \text{ constant.}$$

Consequently

$$f = u + iv = \left(x^2 - y^2\right) + e^{-x} \cos y + i\left(2xy - e^{-x} \sin y\right) + i\alpha_0.$$

But since $z^2 = x^2 - y^2 + 2ixy$ and $\cos y - i \sin y = e^{-iy}$, we get

$$f(z) = z^2 + e^{-z} + i\alpha_0$$

with $\alpha_0 \in \mathbb{R}$.

Exercice 2. Using the Cauchy Theorem and the Cauchy integral formula compute the following integrals:

(1)

$$\int_\Gamma \frac{z^3 + 2z^2 + 2}{z - 2i} dz \quad \text{where} \quad \Gamma = \left\{ z \in \mathbb{C} \,|\, |z - 2i| = \frac{1}{4} \right\}$$

(2)

$$\int_\Gamma \frac{3z^2 + 2z + \sin(z + 1)}{(z - 2)^2} dz \quad \text{where} \quad \Gamma = \{ z \in \mathbb{C} \,|\, |z - 2| = 1 \}$$

Answer:

(1) Using the Cauchy integral formula we have

$$\int_\Gamma \frac{z^3+2z^2+2}{z-2i}dz = 2\pi i(z^3+2z^2+2)\Big|_{z=2i} = 16\pi - 12i$$

(2) Using the Cauchy integral formula we have

$$\begin{aligned}\int_\Gamma \frac{3z^2+2z+\sin(z+1)}{(z-2)^2}dz &= \frac{2\pi i}{1!}(3z^2+2z+\sin(z+1))^{(1)}\Big|_{z=2}\\ &= \frac{2\pi i}{1!}(6z+2+\cos(z+1))\Big|_{z=2} = 2\pi i(14+\cos(3))\end{aligned}$$

Exercice 3. For the following functions, compute the Laurent expansion around z_0 and determine whether z_0 is a regular or singular point; in the latter case, say what is the order of pole at z_0. Recall that if f is holomorphic at z_0 its Laurent expansion is simply the Taylor expansion.

(1) $f(z) = \frac{z}{1+z^2}$ and $z_0 = 1$

(1) $f(z) = \frac{z^2+2z+1}{1+z}$ and $z_0 = -1$

(3) (Bonus) $f(z) = \frac{z^2+z+1}{z^2-1}$ and $z_0 = 1$

Answer:

(1) Since the function $h(z) = z$ is holomorphic in $\mathbb{C}$, $g(z) = \frac{1}{1+z^2}$ is holomorphic except in $z = \pm i$, we can conclude that $z_0 = 1$ is a regular point and therefore Laurent series is equal to Taylor series around z_0. We start by deducing that

$$\begin{aligned}g(z) = \frac{1}{1+z^2} &= \frac{-1}{2i}\left(\frac{1}{i-z}+\frac{1}{i+z}\right)\\ &= \frac{-1}{2i}\left(\frac{1}{(i-1)-(z-1)}+\frac{1}{(i+1)+(z-1)}\right)\\ &= \frac{-1}{2i}\left[\frac{1}{i-1}\cdot\frac{1}{1-\frac{z-1}{i-1}}+\frac{1}{i+1}\cdot\frac{1}{1+\frac{z-1}{i+1}}\right]\end{aligned}$$

Using the formula of Taylor expansion of $\frac{1}{1+x}$ at points $\frac{z-1}{i-1}$ and $\frac{z-1}{i+1}$, we get

$$\begin{aligned}g(z) &= \frac{-1}{2i}\left[\frac{1}{i-1}\sum_{n=0}^{+\infty}\frac{1}{(i-1)^n}(z-1)^n + \frac{1}{i+1}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(i+1)^n}(z-1)^n\right]\\ &= \frac{-1}{2i}\sum_{n=0}^{+\infty}\left(\frac{1}{(i-1)^{n+1}}+\frac{(-1)^n}{(i+1)^{n+1}}\right)(z-1)^n\\ &= \sum_{n=0}^{+\infty}\frac{(-1)^n}{2i}\left[\frac{1}{(1-i)^{n+1}}-\frac{1}{(1+i)^{n+1}}\right](z-1)^n\end{aligned}$$

By noting

$$1 - i = \sqrt{2}e^{-i\frac{\pi}{4}} \quad \text{et} \quad 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$$

The coefficient c_n associated with the term $(z-1)^n$ can be simplified as

$$c_n = \frac{(-1)^n}{2i}\left[\frac{1}{(1-i)^{n+1}} - \frac{1}{(1+i)^{n+1}}\right] = (-1)^n \frac{\sin\left[(n+1)\frac{\pi}{4}\right]}{2^{\frac{n+1}{2}}}$$

Finally

$$\begin{aligned}
f(z) &= zg(z) \\
&= (z-1+1)\sum_{n=0}^{+\infty} c_n(z-1)^n \\
&= \sum_{n=0}^{+\infty} c_n(z-1)^{n+1} + \sum_{n=0}^{+\infty} c_n(z-1)^n \\
&= \sum_{m=1}^{+\infty} c_{m-1}(z-1)^m + c_0 + \sum_{n=1}^{+\infty} c_n(z-1)^n \\
&= c_0 + \sum_{n=1}^{+\infty} (c_{n-1} + c_n)(z-1)^n
\end{aligned}$$

(2)

$$f(z) = \frac{z^2+2z+1}{z+1} = \frac{(z+1)^2}{z+1} = z+1$$

therefore $z_0 = -1$ is a regular point. In this case, the Taylor expansion has the trivial expression $f(z) = z+1$.

(3)

$$f(z) = \frac{z^2+z+1}{z^2-1} = 1 - \frac{1}{2(z+1)} + \frac{3}{2(z-1)}$$

$f(z)$ has two singular points $z = \pm 1$, the convergence happens when $0 < |z-1| < 2$, the Taylor expansion of $\frac{1}{1+z}$ around $z_0 = 1$ is given by

$$\begin{aligned}
\frac{1}{z+1} = \frac{1}{2+(z-1)} = \frac{1}{2}\frac{1}{1+\frac{z-1}{2}} &= \frac{1}{2}\sum_{n=0}^{+\infty}(-1)^n\left(\frac{z-1}{2}\right)^n \\
&= \sum_{n=0}^{+\infty}\frac{(-1)^n}{2^{n+1}}(z-1)^n.
\end{aligned}$$

therefore

$$\begin{aligned}
f(z) &= \frac{3}{2}\frac{1}{z-1} + 1 - \frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{2^{n+1}}(z-1)^n \\
&= \frac{3}{2}\frac{1}{z-1} + \frac{3}{4} + \sum_{n=1}^{+\infty}\frac{(-1)^n}{2^{n+2}}(z-1)^n
\end{aligned}$$

Exercice 4. Compute the following real integrals:

(1)
$$\int_0^{2\pi} \frac{\cos(\theta)\sin(2\theta)}{5+3\cos(2\theta)} d\theta$$

(2)
$$\int_{-\infty}^{+\infty} \frac{x^2}{1+x^6} dx$$

Answer:

(1) We let $z = e^{i\theta}$ and we find

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z} \tag{1}$$

$$\cos(2\theta) = \frac{e^{2i\theta}+e^{-2i\theta}}{2} = \frac{z^4+1}{2z^2} \quad \text{et} \quad \sin(2\theta) = \frac{e^{2i\theta}-e^{-2i\theta}}{2i} = \frac{z^4-1}{2iz^2}. \tag{2}$$

then we define for $f(\cos\theta, \sin\theta) = \cos\theta\sin(2\theta)(5+3\cos(2\theta))^{-1}$

$$\widetilde{f}(z) = \frac{1}{iz} f\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right) = \frac{-\left(z^2+1\right)\left(z^4-1\right)}{6z^2\left(z^2+3\right)\left(z^2+\frac{1}{3}\right)}. \tag{3}$$

We then find that the only singularities inside the unit circle are 0 which is a pole of order 2 and $\pm\frac{i}{\sqrt{3}}$ which are poles of order 1. Their residues are therefore

$$\begin{aligned} \mathrm{Res}_{\frac{i}{\sqrt{3}}}(\tilde{f}) &= \lim_{z\to\frac{i}{\sqrt{3}}} \left[\frac{-\left(z^2+1\right)\left(z^4-1\right)}{6z^2\left(z^2+3\right)\left(z+\frac{i}{\sqrt{3}}\right)}\right] = \frac{i}{6\sqrt{3}} \\ \mathrm{Res}_{\frac{-i}{\sqrt{3}}}(\widetilde{f}) &= \lim_{z\to\frac{-i}{\sqrt{3}}} \left[\frac{-\left(z^2+1\right)\left(z^4-1\right)}{6z^2\left(z^2+3\right)\left(z-\frac{i}{\sqrt{3}}\right)}\right] = -\frac{i}{6\sqrt{3}} \\ \mathrm{Res}_0(\widetilde{f}) &= -\frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left[\frac{z^6+z^4-z^2-1}{3z^4+10z^2+3}\right] = 0. \end{aligned} \tag{4}$$

We find that if γ is the unit circle, then

$$\begin{aligned} \int_0^{2\pi} \frac{\cos\theta\sin(2\theta)}{5+3\cos(2\theta)} d\theta &= \int_\gamma \tilde{f}(z)dz \\ &= 2\pi i\left[\mathrm{Res}_{\frac{i}{\sqrt{3}}}(\tilde{f}) + \mathrm{Res}_{\frac{-i}{\sqrt{3}}}(\tilde{f}) + \mathrm{Res}_0(\tilde{f})\right] = 0. \end{aligned} \tag{5}$$

This result could have been deduced immediately due to the symmetry.

(2) By the triangle inequality, we have, if $z = re^{i\theta}$

$$\left|z^6+1\right| \geq \left||z^6|-1\right| = \left|r^6-1\right|. \tag{6}$$

We write for $r > 1$

$$\begin{aligned}\left|\int_{C_r} \frac{z^2}{1+z^6}dz\right| &= \left|\int_0^\pi \frac{\left(re^{i\theta}\right)^2}{1+\left(re^{i\theta}\right)^6} ire^{i\theta}d\theta\right| \\ &\leq \int_0^\pi \frac{r^3}{r^6-1}d\theta = \frac{\pi r^3}{r^6-1} \to 0 \text{ for } r \to \infty\end{aligned} \tag{7}$$

Let $r > 1$ and $\gamma_r = C_r \cup L_r$, where C_r is defined in the second question and L_r, is the line segment $[-r, r]$ on the real axis. The singularities of $f(2) = \frac{1+z^2}{1+z^6}$ are the zeros of $1 + z^6$. Or

$$1 + z^6 = 0 \Leftrightarrow z^6 = -1 = e^{i(\pi+2\pi n)} \Leftrightarrow z = e^{\frac{i\pi(1+2n)}{6}}, n = 0, 1...5. \tag{8}$$

Only z_0, z_1 and z_2 are inside γ_r and they are poles of order 1. We therefore have that their residues are given using Proposition 11.5 (let $p(z) = z^2$ and $q(z) = 1 + z^6$ which implies $q'(z) = 6z^5$) by

$$\text{Res}_{z_n}\left(\frac{1+z^2}{1+z^6}\right) = \frac{1}{6z_n^3} = e^{-\frac{i\pi(1+2n)}{2}} = \frac{(-1)^{n+1}}{6}i, n = 0, 1, 2. \tag{9}$$

The residue theorem therefore allows us to write

$$\int_{\gamma_r} f(z)dz = \int_{C_r} f(z)dz + \int_{L_r} f(z)dz = 2\pi i \sum_{n=0}^{2} \text{Res}_{z_n}(f) = \frac{\pi}{3}. \tag{10}$$

As

$$\int_{L_r} f(z)dz = \int_{-r}^{r} \frac{x^2}{1+x^6}dx \to \int_{-\infty}^{\infty} \frac{x^2}{1+x^6}dx, \text{ when } r \to \infty \tag{11}$$

(cf. the second question) we deduce that

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^6}dx = \lim_{r\to\infty} \int_{\gamma_r} f(z)dz = \frac{\pi}{3} \tag{12}$$

Exercice 5. With the help of the Table of Laplace transforms, find the Laplace transform of:

$$f(t) = e^{-2t}(3\cos(6t) - 5\sin(6t))$$

Answer: see exercise 4 of session 7

Exercice 6. For $\lambda \in \mathbb{R}$ and y_0, y_1 arbitrary values, find the solutions to:

$$\begin{cases} y''(x) + \lambda y(x) = 0, t > 0 \\ y(0) = y_0 \quad y'(0) = y_1 \end{cases}$$

Answer: Let $Y(z) = \mathfrak{L}(y)(z)$. We know that $\mathfrak{L}(y'')(z) = z^2 Y(z) - zy_0 - y_1$. Thus we have

$$z^2 Y - zy_0 - y_1 + \lambda Y = 0 \quad \Rightarrow \quad Y = \frac{zy_0 + y_1}{z^2 + \lambda} = y_0 \frac{z}{z^2 + \lambda} + y_1 \frac{1}{z^2 + \lambda}.$$

We study the three cases $\lambda = 0, \lambda < 0$ and $\lambda > 0$.

<u>Cas 1</u>: $\lambda = 0$. Then the solution is trivially given by

$$y(x) = y_0 + y_1 x$$

<u>Cas 2</u>: $\lambda < 0$. Let $\lambda = -\mu^2$. Then using the Table of Laplace transforms we get

$$y(x) = y_0 \cosh(\mu x) + \frac{y_1}{\mu} \sinh(\mu x).$$

<u>Cas 3</u>: $\lambda > 0$. Let $\lambda = \mu^2$. Then using the Table of Laplace transforms we get

$$y(x) = y_0 \cos(\mu x) + \frac{y_1}{\mu} \sin(\mu x).$$