Exercise sheet 4

Disclaimer: the exercises are not ordered by increasing difficulty, so you are welcome to work on them in any order that you want.

Spaces of continuous functions II

Exercise 1. *When D is closed and bounded, is the uniform norm the only norm we can put on $(C(D,\mathbb{R}),+)$ such that completeness holds?*

Exercise 2. *Find an example of D that is not closed or not bounded, and $f \in C(D,\mathbb{R})$ such that $\|f\|_\infty$ is infinite.*

Show that if we define a distance $\hat{d}_\infty(f,g) := \min(\|f-g\|_\infty, 1)$ on $C(D,\mathbb{R})$ where $D \subset \mathbb{R}$ is arbitrary, then $(C(D,\mathbb{R}),\hat{d}_\infty)$ is still complete. Argue that $\hat{d}_\infty$ can not be represented by a norm, i.e. there is no norm $\|\cdot\|$ such that $\|f\| = \hat{d}_\infty(0,f)$.

Exercise 3 (Peano curve)**.** *The aims is to construct a surjective continuous map from $([0,1],d_E)$ to $([0,1]^2,d_E)$, i.e. a space-filling curve.*

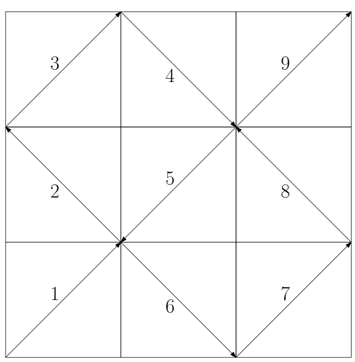

Figure 1: Step 2 of the iteration. The function is to be thought of as going along the squares in the order of the numbers in the picture.

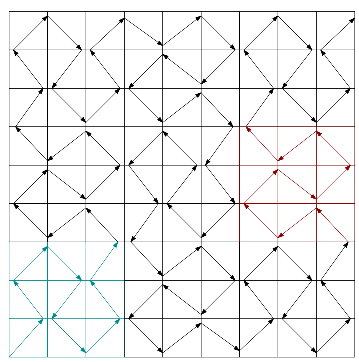

Figure 2: Step 3. The arrows' endpoints are represented slightly shifted for more readability

The idea is to define a sequence $(f_n)_{n\geq 1}$ of almost space-filling curves and to show that this sequence is Cauchy in $(C([0,1],\mathbb{R}^2),d_\infty)$. We define these functions recursively: f_1 is simply the function $x \mapsto (x,x)$. For f_2, divide the unit square $[0,1]^2$ in $\mathbb{R}^2$ in nine squares of length $1/3$ and number them as in Picture 1. On the time interval $[i/9,(i+1)/9]$, $0 \leq i \leq 8$, f_2 draws the line segment in the square i corresponding to the arrow on the picture, where the head of the arrow indicates the endpoint $f_2((i+1)/9)$. For instance, for $x \in [0,1/9]$, $f_2(x) = (3x,3x)$, and for $x \in [1/9,2/9]$, $f_2(x) = (1/3 - 3(x-1/9), 1/3 + 3(x-1/9))$. This defines a continuous function from $[0,1]$ to $[0,1]^2$ with $f_2(0) = (0,0), f_2(1) = (1,1)$. Then, recursively, f_n draws diagonal line segments in a subdivision of $[0,1]^2$ into 3^n smaller squares: at step $n+1$, each such square C is itself divided into nine, and f_{n+1} draws a line segment in each of the nine subsquares, in four possible orders. If f_n goes through C from the bottom-left corner to the top-right corner, then f_{n+1} goes through the nine squares in the order of Fig 1, see the blue part of Fig 2. If f_n goes through C through another corner, the ordering corresponds to the according rotation of Fig 1: for instance if

f_n goes through C by the bottom-right corner to the top-left corner, then the ordering of the nine subsquares is Fig 1 rotated by 180, see the red part of Fig 2.

1. *Show that the sequence $(f_n)_{n\geq 1}$ defined as above is Cauchy, and therefore converges uniformly to a continuous function $f : [0,1] \mapsto [0,1]^2$.*
2. *Show that f is surjective.*

Hint: What can you say of the distance between a point $(x, y) \in [0,1]^2$ and the image of the function f_n, as $n \to \infty$?

Fourier II

Exercise 4. *Consider the function $f : x \mapsto x$ on $[0,1]$.*

1. *Show that the Fourier coefficients of f are $c_0 = 1/2$, $c_n = 0$, $s_n = -1/(\pi n)$ for $n \geq 1$.*
2. *Using that*

$$\sum_{n=1}^{+\infty} \frac{z^n}{n} = -\log(1-z)$$

for $z \in \mathbb{D} \setminus \{1\}$ (the unit disk in $\mathbb{C}$ minus 1), show that $\sum_{n\geq 1}^{+\infty} \sin(2\pi n x)/(\pi n)$ converges and satisfies the identity

$$f(x) = x = \frac{1}{2} - \sum_{n=1}^{+\infty} \frac{\sin(2\pi n x)}{\pi n}.$$

3. *Is the identity still valid at $x = 0$ or $x = 1$?*

Remark 1. *The function f from above, extended by periodicity as $f(x) := f(x - \lfloor x \rfloor)$, is sometimes referred to as the sawtooth function. Its Fourier series doesn't converge at the discontinuity points of f, and actually "overshoots" and produces large peaks around the jumps. This is known as Gibb's phenomenon https://en.wikipedia.org/wiki/Gibbs_phenomenon.*

There are continuous functions f for which the Fourier series, instead of converging at a point x to a value that is different to $f(x)$, actually diverges: but they are harder to come up with. See for instance https://en.wikipedia.org/wiki/Fourier_series#Divergence.