Exercise sheet 3

Disclaimer: the exercises are not ordered by increasing difficulty, so you are welcome to work on them in any order that you want.

Spaces of continuous functions

Exercise 1. *Let $F : C([0,1],\mathbb{R}) \to \mathbb{R}$ be defined by $F(f) := \int_0^1 f(x)dx$, where we consider the Riemann integral. Prove that F is continuous w.r.t. the uniform metric: i.e. show that for any $\varepsilon > 0$, we can find $\delta > 0$ such that if $\|f-g\|_\infty < \delta$, then $|F(f) - F(g)| < \varepsilon$. What does it say if f denotes the density of a line-like object?*

Exercise 2. *Show that the set of functions $f_n : x \mapsto \sin(nx), x \in [0,1]$ defined for all $n \geq 1$ admits no subsequence that converges w.r.t. the norm $|\cdot|_\infty$.*

Fourier

The next two exercises are about proving Lemma 1.8 and 1.11. We remind that the Fourier expansion or Fourier series of a function f on $[0,1]$ corresponds to the writing

$$f(x) = \sum_{n\geq 1} s_n \sin(2\pi nx) + \sum_{n\geq 0} c_n \cos(2\pi nx). \tag{1}$$

Exercise 3. *Prove that the following orthogonality relations hold for integers $m, n \geq 0$:*

1. *Cosine-cosine Orthogonality:*

$$\int_0^1 \cos(2\pi nx)\cos(2\pi mx)\,dx = \begin{cases} 1, & \text{if } n = m = 0, \\ \frac{1}{2}, & \text{if } n = m \neq 0, \\ 0, & \text{if } n \neq m. \end{cases}$$

2. *Sine-sine Orthogonality:*

$$\int_0^1 \sin(2\pi nx)\sin(2\pi mx)\,dx = \begin{cases} 0, & \text{if } n = 0 \text{ or } m = 0, \\ \frac{1}{2}, & \text{if } n = m \neq 0, \\ 0, & \text{if } n \neq m. \end{cases}$$

3. *Sine-cosine Orthogonality:*

$$\int_0^1 \sin(2\pi nx)\cos(2\pi mx)\,dx = 0 \quad \forall n, m.$$

Exercise 4. *Suppose that $f \in C([0,1],\mathbb{R})$ is k times continuously differentiable and satisfies $f^j(0) = f^j(1)$ for all $j = 0 \ldots k-1$ [1] Then prove that there is some $C > 0$ such that for all $n \geq 1$ $|c_n| \leq Cn^{-k}$ and $|s_n| \leq Cn^{-k}$.*

[1] Here by $f^j(x)$ we mean the $j-th$ derivative of f at x, the $0-$th derivative being the function itself.