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Exercise sheet 2

An appetizer to Fourier

Exercise 1. *Consider G the graph given by a cycle of length n, i.e. G is the graph with vertices $V := \{1, \dots, n\}$ and edges $E := \{(i, i+1), i = 1, \dots, n\}$ where we identify $n+1$ to the vertex 1. Let $f : V \to \mathbb{R}$ and consider the discrete heat equation on G with initial data given by f:*

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} &= K\Delta_d u(t,x) \\ u(0,x) &= f(x) \end{cases},$$

where $K > 0$ is the diffusion constant and Δ_d the discrete Laplacian[1] on G:

$$(\Delta_d f)(x) = \frac{1}{2}(f(x+1) + f(x-1) - 2f(x)).$$

Express Δ_d as a matrix and diagonalize it explicitly (by finding the eigenvectors and eigenvalues). Explain how this gives you a solution to the heat equation with the given initial data. What about uniqueness? Check with the example of one initial heat source ($f : x \mapsto \mathbf{1}_{x=1}$) that your solution makes sense physically.

Consider now the same setup on an arbitrary homogeneous graph G, i.e. such that the number of neighbors of any vertex $x \in G$ is a constant $d \in \mathbb{N}$, and with the Laplacian given by

$$(\Delta_d f)(x) = \frac{1}{d}\sum_{y\sim x}(f(y) - f(x)),$$

where the sum is over the neighbors of x in G. Can you rigorously extend the previous argument to this more general framework?

Hint: For the first part, notice that the eigenvectors of the Laplacian must correspond to stationary initial heat configurations!

Exercise 2 (Fourier, approximated in frequency space)**.** *Consider now the heat equation on the circle $\mathbb{S}^1$[2]*

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} &= K\Delta u(t,x) \\ u(0,x) &= f(x) \end{cases},$$

where the initial heat configuration f is given by a trigonometric polynomial[3]:

$$f(x) = b_0 + \sum_{n=1}^{p} a_n \sin(2\pi nx) + \sum_{n=1}^{p} b_n \cos(2\pi nx), \quad b_0, (a_n)_{n=1}^p, (b_n)_{n=1}^p \subset \mathbb{R}.$$

Prove rigorously that in this case the heat equation reduces to a finite system of ODEs. Solve them explicitly to find a solution with initial condition f (without worrying about uniqueness).

What is missing to obtain a solution for any initial $f : \mathbb{S}^1 \to \mathbb{R}$?

[1] For an intuitive explanation of the definition and meaning of this operator, see `https://math.stackexchange.com/questions/50274/intuitive-interpretation-of-the-laplacian-operator`.

[2] Which here we understand as $[0,1]$, with all functions defined with periodic boundary conditions

[3] Equivalently, one may write $f(x) = \sum_{n=-p}^{p} c_n e^{inx}, (c_n)_{n=-p}^{p} \subset \mathbb{C}$.

Remark 1. *Although the solution is relatively easy to find in this set-up, establishing its uniqueness is not. We will come back to this question later in the course, but we invite you to think about it and compare the steps of the derivation of the solution to the set-up of Exercise 1, where uniqueness is clearer.*

Continuous functions

Exercise 3. *Let* $D \subset \mathbb{R}^n$ *be a box*[4]*. Find a sequence of functions* $(f_k)_{k\geq 1} \subset C(D, \mathbb{R})$ *that converges pointwise to a function* f *that is not continuous.*

Exercise 4. *Find all functions* $f \in C([0,1], \mathbb{R})$ *such that the sequence of powers* $(f^n)_{n\geq 1}$ *converges in uniform norm, and characterize the possible limits.*

Exercise 5. *Show that in fact* $C(D, \mathbb{R})$ *has also a multiplicative structure: if* $f, g \in C(D, \mathbb{R})$*, then also the product* $h(x) := f(x)g(x)$ *is in* $C(D, \mathbb{R})$*. What about the function* $\max(f, g)$*?*

Exercise 6. *Recall the axioms of a vector space and verify them in the case of* $(C(D, \mathbb{R}), +)$.

[4] i.e. there exist $(a_i)_{i=1}^n, (b_i)_{i=1}^n, a_i < b_i \ \forall 1 \leq i \leq n$ such that $D = \prod_{i=1}^n [a_i, b_i]$.