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Exercise sheet 9

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

Measurability

Exercise 1 *Recall that a finite simple function is a function $f : \mathbb{R}^n \to \mathbb{R}$ that may be written*

$$f = \sum_{i=1}^{n} c_i \mathbf{1}_{E_i}$$

where $n \geq 1$, $(c_i)_{i=1}^n \subset \mathbb{R}$ and $(E_i)_{i=1}^n \subset \mathbb{R}^n$ are disjoint Borel sets.

Show that for any measurable function $g : \mathbb{R}^n \to \mathbb{R}$, there exists a sequence of finite simple functions converging pointwise to g.

Lebesgue integral

Exercise 2 *Show that $x \mapsto \exp(-|x|)$ is integrable over $\mathbb{R}$, but the constant function c for example is not integrable for $c \neq 0$. Is $f : x \mapsto x$ integrable over $\mathbb{R}$?*

Exercise 3 *Let $f, g : \mathbb{R} \to \mathbb{R}$ be measurable and integrable. Is fg necessarily integrable? What if $|g(x)| \leq 1$ for all $x \in \mathbb{R}$?*

Exercise 4 *In probability/statistics you have seen the notion of a random variable X. Suppose X takes values in the set $C = \{c_1, c_2, \dots\}$, each with probability $\mathbb{P}(c_i)$. What is the expectation (or mean value) of this random variable? Compare this to the Lebesgue integral of a simple function that takes values in the set C.*

Exercise 5 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be integrable and $a \in \mathbb{R}$. Verify that af is integrable and show that $\int af d\lambda = a \int f d\lambda$.*

Hint: You may do it directly from the definition, or use a similar strategy as for linearity: show it first for simple functions and then argue with convergence theorems.

Convergence theorems

Exercise 6 *Let $f_1 \geq f_2 \geq \dots$ be a decreasing sequence of integrable functions, converging pointwise to an integrable function f. Show that $\int f_n \mathrm{d}\lambda \xrightarrow[n\to\infty]{} \int f \mathrm{d}\lambda$.*