Exercise sheet 7

Disclaimer: the exercises are arranged by theme, not by order of difficulty. You may use the result of Exercise 2 for the other exercises.

Reminders

Exercise 1 (f^{-1} is very nice) *Let $f : A \to B$, I_A, I_B some sets of indices, and $(B_i)_{i \in I_B}$ any collection of subsets of B. Recall that the preimage $f^{-1}(B_i)$ is defined as*

$$f^{-1}(B_i) = \{x \in A : f(x) \in B_i\}.$$

Check that when f is bijective, f^{-1} is actually a function. What if f is not injective or not surjective?

Then prove the following identities:

- $f^{-1}(\cup_{i \in I_B} B_i) = \cup_{i \in I_B} f^{-1}(B_i)$;
- $f^{-1}(\cap_{i \in I_B} B_i) = \cap_{i \in I_B} f^{-1}(B_i)$.

Measurable functions

Exercise 2 (Other definitions of measurability) *Prove that a function f is measurable if and only if for all $a < b \in \mathbb{R}$, $f^{-1}((a,b))$ is Borel-measurable. Also prove that the same works if one replaces (a,b) with $[a,b]$.*[1]

Exercise 3 *Show that if f, g are measurable, then so are $f + g$ and fg.*

Exercise 4 *Show that continuous functions are measurable.*

Exercise 5 *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function and $f_n : \mathbb{R} \to \mathbb{R}$ its dyadic approximations given by $f_n(x) = 2^{-n} \lfloor 2^n f(x) \rfloor$. For $m \geq n \geq 1$, show the bound*

$$f(x) - 2^{-n} \leq f_n(x) \leq f_m(x) \leq f(x).$$

Exercise 6 *Prove Lemma 2.14 from the class, by showing first that*

$$f^{-1}([a,b)) = \bigcap_{j \geq 1} \bigcup_{k \geq 1} \bigcup_{n \geq 1} \bigcap_{m \geq n} f_n^{-1}([a - 1/j, b - 1/k))$$

[1] Actually, replacing $(a,b), a, b \in \mathbb{R}$ by $(-\infty, b), b \in \mathbb{R}$, or $(a, +\infty), a \in \mathbb{R}$, or $(a,b], a < b \in \mathbb{R}$, etc... lead to all equivalent definitions.

For fun (non-examinable)

Exercise 7 *Show that there is no collection of intervals* $([a_i, b_i])_{i=1}^n$ *covering* $\mathbb{Q} \cap [0,1]$, *i.e. such that* $\bigcup_{i=1}^n [a_i, b_i] \supseteq \mathbb{Q} \cap [0,1]$, *but having total length* $\sum_{i=1}^n (b_i - a_i) < 1$. *On the other hand, show that it is possible with countably many intervals, i.e. find* $([a_i, b_i])_{i=1}^{+\infty}$ *covering* $\mathbb{Q} \cap [0,1]$ *of total length arbitrarily small.*

Exercise 8 *A general definition of a measurable function is that for* $(\Omega, \mathcal{F})$ *a measure space,* $f^{-1}([a,b)) \in \mathcal{F}$ *for all* $a < b \in \mathbb{R}$.[2]

- *What are measurable functions relative to the measure space* $(\mathbb{R}^n, \mathcal{P}(\mathbb{R}^n))$*?*
- *Prove that*

$$\mathcal{G} = \{E \subset \mathbb{R} : E \text{ is countable or } E^c \text{ is countable}\}$$

  is a σ*-algebra. Is it bigger or smaller than the Borel* σ*-algebra* $\mathcal{F}_B$*?*
- *Intuitively, should there be more measurable functions from* $(\mathbb{R}^n, \mathcal{G})$ *than from* $(\mathbb{R}^n, \mathcal{F}_B)$*, or less? Then determine exactly which are the measurable functions from* $(\mathbb{R}^n, \mathcal{G})$ *to confirm your intuition.*
- *Show that* $\mathcal{G}$ *is generated by singletons in* $\mathbb{R}^n$*, i.e. that it is the smallest* σ*-algebra containing all* $\{x\}, x \in \mathbb{R}^n$.

[2] This definition is natural because it implies that preimages of Borel sets in $\mathbb{R}$ belong to $\mathcal{F}$, i.e. preimages of measurable sets are measurable.