Physics 2nd year – Analysis IV Juhan Aru

Exercise sheet 6

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

Measurable sets and measure spaces

Exercise 1 *Define a measure space / probability space to describe two unrelated fair coin tosses. What assumptions are you making in giving the description? Define a sigma-algebra suitable for studying the situation where one can only ask if the two coins have the same side up, or different sides up.*

Exercise 2 *Let* $(\Omega, \mathcal{F}, \mu)$ *be a measure space. Prove that if* A, B *are measurable sets, then so is also* $A \setminus B := \{a \in A, a \notin B\}$.

Exercise 3 *Show that the Borel* $\sigma-$*algebra on* $\mathbb{R}^n$ *also contains all products of half-lines* $\Pi_{i=1}^n(-\infty, a_i]$, *all open balls* $B(x,r)$ *and in fact all open sets of* $\mathbb{R}^n$.

Measures

Exercise 4 *Let* $(\Omega, \mathcal{F}, \mu)$ *be a measure space. Prove that if* $A_1 \subseteq A_2 \subseteq A_3 \dots$ *are an increasing sequence of measurable sets, then* $\mu(\bigcup_{i\geq 1} A_i) = \lim_{i\to\infty} \mu(A_i)$.

Prove also that if $A_1, A_2, \dots$ *are any measurable sets, then the so called union bound* $\mu(\bigcup_{i\geq 1} A_i) \leq \sum_{i\geq 1} \mu(A_i)$ *holds. Interpret it in the probabilistic context.*

Exercise 5 *Show that the Lebesgue measure of* $\mathbb{R}^n$ *is infinite and that the Lebesgue measure of a line segment* $[0,1] \subseteq \mathbb{R}^n$ *is zero.*

Now consider the Lebesgue measure on $\mathbb{R}$. *Prove that the measure of irrational numbers contained in* $[0,R]$ *is equal to* R; *prove also that the Lebesgue measure of the Cantor set is zero.*

Exercise 6 *Show that there is no finite measure on* $(\mathbb{N}, \mathcal{P}(\mathbb{N}))$ *that is translation-invariant, i.e. such that* $\mu(A+n) = \mu(A)$ *for all* $n \in \mathbb{N}$ *and* $A \in \mathcal{P}(\mathbb{N})$.

For fun (non-examinable)

Exercise 7 (Borel $\sigma-$algebra) *Let* Ω *and* I *be two non-empty sets. Suppose that for each* $i \in I$, $\mathcal{F}_i$ *is a* $\sigma-$*algebra on* Ω.

- *Prove that* $\mathcal{F} := \bigcap_{i\in I} \mathcal{F}_i$ *is also a* $\sigma-$*algebra on* Ω.
- *Now, let* $\mathcal{G}$ *be any subset of* $\mathcal{P}(\Omega)$. *Show that there exists a* $\sigma-$*algebra that contains* $\mathcal{G}$ *and that is contained in any other* σ*-algebra containing* $\mathcal{G}$. *This is called the* $\sigma-$*algebra generated by* $\mathcal{G}$.
- *Conclude that the Borel* $\sigma-$*algebra is well-defined.*

Exercise 8 (Non-existence of probability measures on the power set) *There is no measure* μ *on* $(\mathbb{R}, \mathcal{P}(\mathbb{R}))$ *that is translation invariant, i.e. such that for any* $A \in \mathcal{P}(\mathbb{R})$, $\alpha \in \mathbb{R}$, $\mu(A+\alpha) = \mu(A)$, *and locally finite, i.e. such that* $\mu([0,1]) < +\infty$.

Hint: Consider the equivalence relation $x \sim y$ iff $x - y \in \mathbb{Q}$. Use the axiom of choice to construct a set of representatives of equivalence classes and prove by contradiction that this set cannot be measurable.

Remark 1 *Without the axiom of choice*[1]*, one actually cannot prove–nor disprove!–the existence of a non-measurable set. But without the axiom of choice, one cannot disprove either that* $\mathbb{R}$ *is not a countable union of countable sets...*

[1] Recall that the Axiom of choice says the following: if you are giving any collection of non-empty sets $(X_i)_{i \in I}$, then their product is non-empty. In other words, you can define a function $f : I \to \bigcup_{i \in I} X_i$ such that for all $i \in I$, $f(i) \in X_i$.