Exercise sheet 5

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

Continuous functions

Exercise 1 *Find a sequence* $(f_n)_{n\geq 1} \subset C^1([0,1],\mathbb{R})$ *that converge to some* $f \in C(D,\mathbb{R})$ *with respect to the uniform norm* $\|\cdot\|_\infty$*, but where* f *is not differentiable. Now show that if convergence holds w.r.t. the norm* $\|f\|_\infty + \|f'\|_\infty$*, then the limit is also continuously differentiable.*

Riemann integral

Exercise 2 *Show that the Riemann integral satisfies some desirable properties:*

- *All continuous functions on* $[0,1]$ *are Riemann integrable*
- *Every piecewise constant function is Riemann integrable*
- *Linearity: If* f,g *are Riemann integrable on* $[0,1]$*, then so is their sum and the integral is equal to the sums.*

Fourier

Exercise 3 *We aim to conclude the proof of Proposition 1.9 in the notes. Recall that the setup consisted of* $f \in C^2([0,1],\mathbb{R})$ *twice continuously differentiable and satisfying* $f(0) = f(1)$ *and* $f'(0) = f'(1)$*, and that we argued in the first part of the proof that*

$$f_N := \sum_{n=1}^{N} (s_n \sin(2\pi n x) + c_n \cos(2\pi n x)),$$

converges w.r.t. $\|\cdot\|_\infty$ *as* $N \to \infty$ *to some function* $g \in C([0,1],\mathbb{R})$*, where* $(s_n)_{n\geq 1}$ *and* $(c_n)_{n\geq 1}$ *are the Fourier coefficients of* f*.*

Using the definition of f_N *and* g *show that for all* $n \geq 0$*,*

$$\int_0^1 (f-g)\cos(2\pi n x)\mathrm{d}x = \int_0^1 (f-g)\sin(2\pi n x)\mathrm{d}x = 0,$$

and conclude the proof with the help of Proposition 1.12.

Exercise 4 (Fejér kernel, I) *One possible choice for the function* T_{n,x_0} *in the notes is the so called Fejér kernel, denoted* $F_n^{x_0}$*. The Fejér kernel for* $x_0 = 0$ *is given by*

$$F_n^0(x) = 1 + \sum_{k=1}^{n-1} 2\left(1 - \frac{k}{n}\right)\cos(2\pi k x).$$

Deduce the expression for $F_n^{x_0}$ *for* $x_0 \in (0,1)$*.*

Now prove carefully the following properties, thereby proving Lemma 1.13:

1. $\forall n \geq 1,\ F_n^{x_0}(x) \geq 0$,
2. $\forall n \geq 1,\ \int_0^1 F_n^{x_0}(x)\mathrm{d}x = 1$,

3. $\forall \varepsilon > 0, \lim_{n\to\infty} \int_0^1 \mathbf{1}_{|x-x_0|>\varepsilon} F_n^{x_0}(x)\mathrm{d}x = 0.$

Hint: For 1., rewrite the cosine in the expression of F_0^n with complex exponentials and prove the identity

$$\sum_{k=-n+1}^{n-1} (n-|k|)e^{2\pi ikx} = (e^{2\pi ix\frac{n-1}{2}} + e^{2\pi ix\frac{n-3}{2}} + \cdots + e^{-2\pi ix\frac{n-1}{2}})^2$$

Exercise 5 (Fejér kernel, II)

1. *For $f \in C([0,1],\mathbb{R})$ such that $f(0) = f(1)$, show that*

$$\int_0^1 f(x)F_n^{x_0}(x)\mathrm{d}x \xrightarrow[n\to\infty]{} f(x_0).$$

Therefore, the Fejér kernel is in some sense an 'approximate Dirac delta function'.

2. *Compute formally the Fourier coefficients of the Dirac delta function, i.e. a 'function'*[1] *$\delta_{x_0} : [0,1] \to \mathbb{R}^+$ for $x_0 \in [0,1]$ fixed such that for all $f \in C([0,1],\mathbb{R})$ such that $f(0) = f(1)$,*

$$\int_0^1 f(x)\delta_{x_0}(x)\mathrm{d}x = f(x_0),$$

and write formally its Fourier series.

3. *Show that the Fourier coefficients of $F_n^{x_0}$ converge to the formal expression that you found in 2.*

Cantor set

Exercise 6 (The Cantor set) *Consider the following iteration: we set $C_0 = [0,1]$ and obtain C_1 by removing the middle third, i.e. $C_1 = C_0 \setminus (1/3, 2/3)$. Now to obtain C_2, we remove the middle third of the both remaining intervals. We continue iteratively and define $C = \cap_{i\geq 1} C_i$. Prove that C is a closed set (i.e. $[0,1]\setminus C$ is open) with empty interior (i.e. there is no open interval contained in C).*

It also has uncountably many elements and is a perfect set - both of these are in the for fun section. It can also be described as the set of numbers in $[0,1]$ that admit a representation in ternary expansion containing no occurrence of 1.

For fun (non-examinable)

Exercise 7 (The Cantor set is perfect) *Prove that the Cantor set C has uncountably many elements and is what is called a perfect set - it has no isolated points in the sense that for each point $x \in C$ there is a sequence of elements $x_n \in C$ with $x_n \neq x$ that converges to x.*

[1] Such a function does not exist, in the classical sense!