Exercise sheet 14

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

More Fourier series and transforms

Exercise 1 (Legendre polynomials)**.** *The goal of this exercise is to show that the Gram-Schmidt orthogonalization procedure applied to the sequence of monomials* $(x \mapsto x^n)_{n\geq 0}$ *on* $L^2([-1,1])$ *yields the (normalized) Legendre polynomials* $(\sqrt{n+1/2}P_n)_{n\geq 0}$*, where*

$$(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x), \quad P_0 = 1, P_1(x) = x.$$

1. *Show that the polynomial obtained at the* n*-th step of Gram-Schmidt is given by*

$$p_n(x) = xp_{n-1}(x) - \frac{\langle p_{n-1}, xp_{n-1}\rangle}{\langle p_{n-1}, p_{n-1}\rangle}p_{n-1}(x) - \frac{\langle p_{n-2}, xp_{n-1}\rangle}{\langle p_{n-2}, p_{n-2}\rangle}p_{n-2}(x).$$

   It might be helpful to first show that the projection of the monomial x^n *onto the span of* $\{p_k\}_{k=1}^{n-1}$ *is the same as that of* xp_{n-1}*.*

2. *Show by induction that* p_n *is an even function (i.e.* $p_n(x) = p_n(-x)$*) for* n *even, and an odd function (i.e.* $p_n(x) = -p_n(-x)$*) when* n *is odd.*

3. *Conclude that* $p_n = P_n \cdot p_n(1)$*, by showing that they satisfy the same recurrence relation.*

Exercise 2. *In this exercise we aim to show that* $L^2(\mathbb{R})$ *is separable.*

- *Prove that each* $f \in L^2(\mathbb{R})$ *can be approximated arbitrarily well by* $f1_{[-n,n]}$ *for* n *large enough, meaning that for every* $\varepsilon > 0$ *there is some* $n \in \mathbb{N}$ *with* $\|f - f1_{[-n,n]}\|_2 < \varepsilon$*.*
- *By using dense countable subsets of* $L^2([-n,n])$ *(which we know to be separable), find a countable dense subset of* $L^2(\mathbb{R})$*.*

[1].

Exercise 3 (Heisenberg uncertainty principle)**.** *Let* $f : \mathbb{R} \to \mathbb{C}$ *be a smooth function such that all its derivatives decay more than polynomially fast - more precisely, such that* $x^m f^{(n)}$ *are bounded for any* $m, n \in \mathbb{N}$ *with* $f^{(n)}$ *denoting the* $n-$*th derivative (this is called a Schwartz function). Suppose further that* $\int_{\mathbb{R}} |f^2| d\lambda = 1$*. Show that*

- $\int_{\mathbb{R}} |\hat{f}^2| d\lambda = 1$*;*
- *both* $x^2|f(x)|^2$ *and* $k^2|\hat{f}(k)|^2$ *are integrable;*

  Hint: look at the Fourier transform of f'*.*

- *the following uncertainty principle holds:*

$$\left(\int_{\mathbb{R}} x^2|f(x)|^2 d\lambda(x)\right)\left(\int_{\mathbb{R}} k^2|\hat{f}(k)|^2 d\lambda(x)\right) \geq \frac{1}{16\pi^2}.$$

[1]Why is $\{\exp(2\pi i k x)\}_{k\in\mathbb{R}}$ not a basis?
Actually, a basis can be neatly constructed from Hermite polynomials and the Hamiltonian of the quantum harmonic oscillator, as studied in Exercise 8.

- *In fact for any x_0, k_0:*

$$\left(\int_{\mathbb{R}} (x_0 - x)^2 |f(x)|^2 d\lambda(x)\right) \left(\int_{\mathbb{R}} (k_0 - k)^2 |\hat{f}(k)|^2 d\lambda(x)\right) \geq \frac{1}{16\pi^2}.$$

Intuitively this says that for any function both f and $\hat{f}$ cannot be simultaneously localised. The interpretation in the realm of quantum mechanics is that the position of the particle and its momentum cannot be localised simultaneously.

Hint: look at the function $x \mapsto \alpha x f + \beta f'(x)$ and expand its L^2 norm.

Operators

Exercise 4. *Consider $(\mathcal{H}, \|\cdot\|)$ a separable Hilbert space and T a bounded linear operator.*

- *Show the following inequality for $f \in H$:*

$$\|Tf\| \leq \|T\|_{op} \|f\|.$$

- *Show that T is continuous in the sense that if a sequence $(f_n)_{n\geq 1}$ converges to f w.r.t $\|\cdot\|$, then also Tf_n converges to Tf.*

Exercise 5 (Boundedness of operators)**.** *We aim to argue that the position operator formally given by $f \to xf$ is not bounded on $L^2(\mathbb{R})$:*

- *Find a square-integrable function f such that $xf(x)$ is not square integrable.*
- *Show that the position operator is well defined for functions f that are square integrable and such that also xf is square integrable.*
- *Find for $i \geq 1$ functions $f_i \in L^2(\mathbb{R})$ of unit norm with $xf_i \in L^2(\mathbb{R})$ but $\|xf_i\|_2 \to \infty$.*

Exercise 6 (Finite rank linear operators)**.** *Consider a real Hilbert space $\mathcal{H}$. Let $u_1, \ldots, u_n \in \mathcal{H}$, $v_1, \ldots, v_n \in \mathcal{H}$, and define $T(f) := \sum_{i=1}^n \langle f, u_i\rangle v_i$.*

- *Prove that T is bounded.*

Now let T be of finite rank and Hermitian, i.e. $\langle Tf, g\rangle = \langle f, Tg\rangle$ for any $f, g \in \mathcal{H}$.

- *Argue that T can be diagonalized, i.e. that we can find $g_1, \ldots g_m$ orthonormal with $m \leq n$ and $\lambda_i \in \mathbb{R}$ such that if we write $f = \sum_{i=1}^m c_i \langle f, g_i\rangle g_i + f_0$ with f_0 orthogonal to $g_1, \ldots, g_m$ then*

$$T(f) = \sum_{i=1}^m c_i \lambda_i \langle f, g_i\rangle g_i.$$

Non-examinable

Exercise 7 (Density of polynomials in L^2)**.** *Show that the set of Legendre polynomials $\{P_n\}_{n\geq 1}$ defined in Exercise 1 is an orthonormal basis of $L^2([-1,1])$, by showing that it is spanning.*

Hint: using the fact that $\{\cos(\pi n x)\}_{n\geq 1}$ is an orthonormal basis of $L^2([0,1])$ and expand $\cos(\pi n x)$ as a polynomial in $\cos(\pi x)$.

Exercise 8. *In this exercise we aim to show that the set of nice functions seen in class is dense in $L^2(\mathbb{R})$. Here a function f was called nice if $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ with support in some $[-m,m]$ and whose Fourier transform satisfies $|\hat{f}(k)| \leq Ck^{-2}$ for all $k \geq 1$ is dense in $L^2(\mathbb{R})$.*

To do this

- *By using the heat equation with initial data $u_0(x) = f(x)$, show that every $f \in L^2([-n,n])$ can be arbitrarily approximated by a function $g \in L^2([-n,n])$ whose Fourier series decays faster Ck^{-2}*
- *Using this show that every $f \in L^2(\mathbb{R})$ can be approximated by a nice function.*

Exercise 9 (Quantizing the harmonic oscillator)**.** *The quantum version of a one-dimensional harmonic oscillator of angular frequency ω and mass m is described by the following Hamiltonian operator on $L^2(\mathbb{R})$:*

$$H := \frac{1}{2m}(P^2 + \omega^2 m^2 X^2),$$

where P is the momentum operator $Pf = -i\hbar f'$ and X the position operator $Xf : x \mapsto xf(x)$. This operator is not bounded on $L^2(\mathbb{R})$: it is only defined on the subset of functions such that xf and f' are integrable: but for simplicity, we will see it as an operator on the Schwartz space $\mathcal{S}$ (see the previous exercise sheet for a definition thereof). The goal of this exercise is to show that the Hamiltonian can still be diagonalized in an appropriate way.

1. *For $f \in \mathcal{S}$, show the commutation relation $[X,P]f = i\hbar f$. We formally write this identity as $[X,P] = i\hbar\mathrm{id}$. Deduce that if one denotes*

   $$a = \alpha X + i\beta P, \quad a^\dagger = \alpha X - i\beta P$$

   with

   $$\alpha = \sqrt{\frac{\omega m}{2\hbar}}, \quad \beta = \sqrt{\frac{1}{2\omega m\hbar}},$$

   then the Hamiltonian can be rewritten $H = \omega h(a^\dagger a + \frac{1}{2}\mathrm{id})$ (understanding this equality as tested against $f \in \mathcal{S}$).

2. *Prove that H is positive semi-definite, i.e. that all its eigenvalues are non-negative. On the other hand, observe that if ϕ is an eigenfunction of eigenvalue λ for H, then $a\phi$ is an eigenfunction with eigenvalue $\lambda - \omega\hbar$ for H or it is zero. Deduce that the lowest eigenvalue for H is $\omega\hbar/2$ and that an eigenfunction having this eigenvalue is in the kernel of a.*

3. *Find a function ϕ_0 in the kernel of a (i.e. an eigenfunction of eigenvalue zero) of norm one and use it to construct eigenfunctions of H of higher eigenvalues.*

4. *Prove by induction that the n-th eigenfunction is of the type $P_n(x)\phi_0(x)$, where P_n is a polynomial of degree n^2. Deduce that the basis of eigenfunctions of H is also a basis of $L^2(\mathbb{R})$, i.e. it is dense in $L^2(\mathbb{R})$.*

The n-th eigenvalue is often referred to in the physics litterature as the state having « n quanta of excitation ». It is interesting to think about why the state with lowest energy has no quantum of excitation, but <u>*does not*</u> *have zero energy (its energy is actually $\omega\hbar/2$).*

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[2] It can actually be computed explicitly and related to Hermite polynomials.