Exercise sheet 13

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

Exercise 1 (Fourier series on larger intervals)**.** *Using Fourier series on* $[0,1]$ *and scaling / translation show that for any* $L \in \mathbb{N}$*, every* $f_L \in L^2([-L/2, L/2])$ *can be written as*

$$f_L(x) = L^{-1} \sum_{n\in\mathbb{Z}} \hat{f}_L(n/L) \exp\big(2\pi i L^{-1} n x\big), \tag{1}$$

where the summing is absolute for any $x \in [-L/2, L/2]$ *and the limit in the series is with respect to the* L^2 *norm, and*

$$\hat{f}_L(n/L) := \int_{[-L/2,L/2]} f_L(x) \exp\big(-2\pi i L^{-1} n x\big) d\lambda(x). \tag{2}$$

Deduce Lemma 3.22, i.e. that the set of functions: $(\sqrt{\frac{2}{L}} \sin\big(\frac{2}{L}\pi n x\big))_{n\geq 1}, (\sqrt{\frac{2}{L}} \cos\big(\frac{2}{L}\pi n x\big))_{n\geq 1}$ *together with the constant function* $\frac{1}{\sqrt{L}}$ *forms an orthonormal basis of* $L^2([-L/2, L/2))$.

Exercise 2 (Finishing Theorem 3.23)**.**

1. *In the set-up of Theorem 3.23, prove rigorously that the three following functions*

$$u_P(t,x) = \langle u_0, 1\rangle + \sum_{n\geq 1} \exp\big(-D4\pi^2 n^2 t\big) \left(\sin(2\pi n x)\langle u_0, 2\sin(2\pi n x)\rangle + \cos(2\pi n x)\langle u_0, 2\cos(2\pi n x)\rangle\right)$$

$$u_D(t,x) = \sum_{n\geq 1} \exp\big(-D\pi^2 n^2 t\big) \sin(\pi n x)\langle u_0, 2\sin(\pi n x)\rangle$$

$$u_N(t,x) = \langle u_0, 1\rangle + \sum_{n\geq 1} \exp\big(-D\pi^2 n^2 t\big) \cos(\pi n x)\langle u_0, 2\cos(2\pi n x)\rangle$$

   are well-defined and belong to $L^2([0,1])$*. Furthermore, prove that for each* $t > 0$ *they are differentiable in* t *and twice differentiable in* x *such that the derivatives are Riemann integrable. Finally, show that they solve the heat equation.*

   Hint: You can use the following result for derivatives of series: let $(f_n)_{n\geq 1} \subset C^1([0,1|, \mathbb{R})$ *be such that the series* $\sum_{n=1}^{+\infty} f_n$ *and* $\sum_{n=1}^{+\infty} f_n'$ *each converge pointwise absolutely to a bounded function. Then* $\sum_{n=1}^{+\infty} f_n \in C^1([0,1], \mathbb{R})$ *with derivative* $\sum_{n=1}^{+\infty} f_n'$.

2. *Finish the proof of uniqueness in Theorem 3.23 by verifying the steps in the formal calculation. In particular, when* $u(t,x)$ *is one of the three functions above argue the following.*

   - *By using the connection to the Riemann integral and known results in that case or otherwise show that for all* $t > 0$,

$$\frac{\partial \|u(t,x)\|^2}{\partial t} = 2\int_{[0,1]} u(t,x) \frac{\partial u(t,x)}{\partial t} d\lambda(x)$$

   - *Similarly show that the integration by parts is allowed:*

$$\int_{[0,1]} u(t,x) \frac{\partial^2 u(t,x)}{\partial x^2} d\lambda(x) = -\int_{[0,1]} \left(\frac{\partial u(t,x)}{\partial x}\right)^2 d\lambda(x).$$

Exercise 3 (Fourier transform of Gaussian density)**.** *The aim of this exercise is to calculate the Fourier transform of the Gaussian density* $\exp\bigl(-x^2/2\bigr)$*.*

- *By allowing yourself to change the order of differentiation / integration and using integration by parts, find a first-order ODE satisfied by the Fourier transform*

$$\hat{f}(k) := \int_{\mathbb{R}} \exp\bigl(-x^2/2\bigr) \exp(-2ki\pi x) d\lambda(x).$$

- *Justify carefully the change of integration and differentiation and integration by parts in part 1.*
- *Solve this ODE and find thus the Fourier transform of the Gaussian density.*
- *Is there a function* $f : \mathbb{R} \to \mathbb{R}$ *that is equal to its Fourier transform?*

Exercise 4 (Convolutions, II)**.** *Recall for* g *a bounded measurable function on* $\mathbb{R}$ *the convolution product* $f \star g$ *on* $L^1(\mathbb{R})$ *(i.e. for* $f \in L^1$*), defined by*

$$(f \star g)(x) := \int_{\mathbb{R}} f(y) g(x-y) \mathrm{d}y.$$

Suppose now that g *is also integrable, i.e. that* $g \in L^1(\mathbb{R})$*. Show that the following identity holds:*

$$\mathcal{F}(f \star g) = \mathcal{F}(f) \cdot \mathcal{F}(g),$$

where $\cdot$ *stands for pointwise multiplication.*

Non-examinable

Exercise 5. *Prove Lemma 3.20 and thus establish that the span of* $(\sin(2\pi n \cdot), \cos(2\pi n \cdot), 1)_{n \geq 1}$ *is equal to* $L^2([0,1])$*.*

You may for example try to argue again using the Féjér kernel, like we did in the case of continuous functions.