# Exercise sheet 12

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

**Exercise 1** (Markov inequality)**.** *Let $f$ be integrable and non-negative and let $\beta > 0$. Prove that $\lambda(\{x : f(x) > \beta\}) \leq (\int f d\lambda)/\beta$.*

*Conclude that if $f, g \in L^1(E)$ satisfy $\|f - g\|_1 \leq \varepsilon$, then $\lambda(\{x : |f(x) - g(x)| > \lambda\}) \leq \varepsilon/\lambda$.*

**Exercise 2.** *Consider the form $\langle \cdot, \cdot \rangle$ on $L^2([0, 1], \mathbb{C})$ defined for $f, g \in L^2([0, 1], \mathbb{C})$ by*

$$\langle f, g \rangle = \int_{[0,1]} f \, \overline{g} \, \mathrm{d}\lambda,$$

*is a (hermitian / complex) scalar product*[1]*, turning $L^2([0, 1], \mathbb{C})$ into a complex vector space. Then, similarly to Exercise 3, sheet 3, show that $(\exp(2\pi i n \cdot))_{n\in\mathbb{Z}}$ are orthonormal functions in $L^2([0, 1])$.*

**Exercise 3.** *Show that, on an inner product space $(V, \langle \cdot, \cdot \rangle)$, the application $v \mapsto \sqrt{\langle v, v \rangle}$ always defines a norm.*

**Exercise 4.** *Let $v_1, v_2, \ldots$ be orthonormal vectors in a complete inner product space $V$. Show that for any $w \in V$, we have that $\widehat{w} := \sum_{i\geq 1} \langle v_i, w \rangle v_i$ is well-defined and satisfies 1) $\|\widehat{w}\| \leq \|w\|$ and 2) $\langle w - \widehat{w}, v_i \rangle = 0$ for all $i \geq 1$.*

**Exercise 5.**

1. *Show that in any inner product space $(V, \langle \cdot, \cdot \rangle)$ with orthonormal basis $(v_i)_{i\geq 1}$, for any $w \in V$ the norm*
$$\|w - \sum_{i=1}^{n} c_i v_i\|$$
*is (strictly) minimized by $c_i = \langle v_i, w \rangle$.*

2. *Using this, show the second item of Lemma 3.16, i.e. that for $(V, \langle \cdot, \cdot \rangle)$ an inner product space admitting an orthonormal basis $(v_i)_{i\geq 1}$, the writing $w = \sum_{i\geq 1} a_i v_i$ of any $w \in V$ is such that each $a_i$ is uniquely determined, and actually equal to $\langle v, v_i \rangle$.*

**Exercise 6.** *Let $l^2(\mathbb{N})$ denote the set of all real-valued sequences $\bar{c} = (c_i)_{i\geq 1}$ with $\sum_{i\geq 1} c_i^2 < \infty$. Show that equipping it with (coordinate-wise) addition and inner product $\langle \bar{a}, \bar{b} \rangle = \sum_{i\geq 1} a_i b_i$ turns it into an inner product space.*

## Non-examinable

**Exercise 7.** *Finish the proof that $L^1$ is complete.*

[1] For positive-definiteness of the scalar product, one would need to identify functions in $L^2([0, 1], \mathbb{C})$ that are equal almost everywhere, like in the real case.