Exercise sheet 10

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

Exercise 1 (Some properties, revisited)**.** *Recall that a property P is said to hold almost everywhere in $\mathbb{R}$, or for almost all $x \in \mathbb{R}$, if the set of $x \in \mathbb{R}$ for which P does not hold is measurable and of Lebesgue measure zero. Show the following form of linearity:*

Theorem. *Let f, g, h be measurable, integrable and suppose that $h = f + g$ holds almost everywhere. Then h is integrable and*

$$\int h d\lambda = \int f d\lambda + \int g d\lambda.$$

Show also the following stronger formulation of the monotone convergence theorem:

Theorem. *Let $(f_n)_{n\geq 1}$ be a sequence of positive and integrable functions from $\mathbb{R}$ to $\mathbb{R}^+$, that is almost everywhere increasing, i.e. such that for all $n \geq 1$, almost everywhere[1] $f_n \leq f_{n+1}$. Suppose also that there exists f measurable such that $f_n \xrightarrow[n\to\infty]{} f$ almost everywhere. Then*

$$\lim_{n\to\infty} \int f_n \mathrm{d}\lambda = \int f \mathrm{d}\lambda.$$

Exercise 2 (Reminder: switching sums)**.** *Provide examples of double sequences $(a_{n,m})_{n,m\in\mathbb{N}}$ such that one of the limits below converges, but not the others; or that they all converge but to different limits:*

$$1) \quad \sum_{n=1}^{+\infty} \left(\sum_{m=1}^{+\infty} a_{n,m} \right) := \lim_{N\to\infty} \sum_{n=1}^{N} \left(\lim_{M\to\infty} \sum_{m=1}^{M} a_{n,m} \right)$$

$$2) \quad \sum_{m=1}^{+\infty} \left(\sum_{n=1}^{+\infty} a_{n,m} \right) := \lim_{M\to\infty} \sum_{m=1}^{M} \left(\lim_{N\to\infty} \sum_{n=1}^{N} a_{n,m} \right)$$

$$3) \quad \lim_{K\to\infty} \sum_{m=1}^{K} \sum_{n=1}^{K} a_{n,m}$$

On the other hand, prove that if $(a_{n,m})_{n,m\in\mathbb{N}}$ is absolutely summable, i.e. if one of these limits above exists when replacing $a_{m,n}$ with $|a_{m,n}|$, then the others do as well and all the results are the same.

[1] Think about whether it is important that it is "for all $n \geq 1$, for almost all $x \in \mathbb{R}$ $f_n(x) \leq f_{n+1}(x)$" or "for almost all $x \in \mathbb{R}$, for all $n \geq 1$, $f_n(x) \leq f_{n+1}(x)$"

Exercise 3. *Let* $f(x_1, \dots, x_n) : \mathbb{R}^n \to \mathbb{R}$ *be (Borel-)measurable. Then for any* $0 < m < n$ *and any* $(x_1, \dots, x_m) \in \mathbb{R}^m$*, we have that* $f_{x_1,\dots,x_m} : \mathbb{R}^{n-m} \to \mathbb{R}$*,* $f_{x_1,\dots,x_m}(y_1, \dots, y_{n-m}) := f(x_1, \dots, x_m, y_1, \dots, y_{n-m})$ *is also measurable.*

Exercise 4. *Show that* $f : (0, 1) \to \mathbb{R}$*,* $x \mapsto x^\alpha$ *is integrable if and only if* $\alpha > -1$*. What about* $f : (1, +\infty) \to \mathbb{R}$*,* $x \mapsto x^\alpha$*? Revisit the Example 2.41 in the notes of finding a function* $f : (-1, 1)^2 \to \mathbb{R}$ *that is integrable, but so that there is some point* $x \in (-1, 1)$ *for which* $f(x, \cdot) : (-1, 1) \to \mathbb{R}$ *is not integrable.*