What will the exam be like?

Model exam questions

In the exam there will be around 4-5 questions in the following style. You can hand in solutions during the exercise session of April 28th or in some other way before the end of 30th April.[1].

Exercise 1 (Measure spaces and Lebesgue measure)**.**

1. *Give a definition of a measure space. Define the Lebesgue measure λ on $\mathbb{R}^2$, defining also all the relevant notions.*
2. *Now consider the function $f(x, y) := \sin(\pi x)\sin(\pi y)$. Explain why the set $\{(x, y) \in \mathbb{R}^2 | f(x, y) = 0\}$ is (Borel-)measurable. Calculate its Lebesgue measure.*
3. *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Prove the union bound: for $E_1, E_2, \dots \in \mathcal{F}$, we have that $\mu(\cup_{i\geq 1} E_i) \leq \sum_{i\geq 1} \mu(E_i)$.*
4. *Show that $\{x : \sin(\pi x) \in \mathbb{Q}\}$ is measurable and calculate its Lebesgue measure.*

Exercise 2 (The Lebesgue integral)**.**

1. *Give a definition of the Lebesgue integral of a measurable function on $\mathbb{R}$.*
2. *Give an example of a non-negative integrable function on $\mathbb{R}$, whose square-root is not integrable.*
3. *State the linearity property for integrable functions defined on $\mathbb{R}$. Now assume that it holds for all simple non-negative functions taking only finitely many values, and prove that it holds for all non-negative integrable functions. You may use any convergence theorems, once properly stated.*
4. *Let $f : \mathbb{R} \to \mathbb{R}$ be an integrable function. Is it true that $f(x) \to 0$ as $x \to \infty$? Give a proof or a counterexample. What about if we assume in addition that f is continuous?*

And possibly there will be 1-2 fun questions too, that are not necessarily difficult but that you should attack after finishing the main questions and that will only come into play when you are aiming for the highest grades.

Exercise 3 (Bonus)**.**

Let f be defined on $[0, 1]$ as $f(x) = 0$ if x is rational, $f(x) = 1/a$ if x is irrational and a is the first non-zero integer in the decimal representation of x. Prove that f is measurable, integrable, and compute $\int_0^1 f \mathrm{d}\lambda$.

[1] We will try to provide corrections, granted that we find enough assistants who are ready to help. If not, written solutions will be provided.