Reminder sheet

1 $\mathbb{R}^n$ as a Euclidean and metric space

Definition 1 (Distance)**.** *Let X be a set. A distance on X is a function $d : X \times X \to \mathbb{R}$ satisfying:*

- *(Positiveness) $\forall x, y \in X$, $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.*
- *(Symmetry) $\forall x, y \in X$, $d(x, y) = d(y, x)$.*
- *(Triangular inequality) $\forall x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.*

Definition 2 (Norm)**.** *Let V be a vector space over $\mathbb{R}$ or $\mathbb{C}$. A norm on X is a function $\|\cdot\| : X \to \mathbb{R}^+$ satisfying:*

- *(Positivity) $\forall v \in V$, $\|v\| = 0$ implies $v = 0$.*
- *(Absolute homogeneity) $\forall v \in V$ and $\lambda \in \mathbb{R}$ or $\mathbb{C}$, $\|\lambda v\| = |\lambda| \|v\|$.*
- *(Triangular inequality) $\forall x, y \in V$, $\|x + y\| \leq \|x\| + \|y\|$.*

Notice that the first and second properties imply that $\|x\| = 0 \iff x = 0$.
Euclidean space: $\mathbb{R}^n$ is an Euclidean space (over $\mathbb{R}$) of dimension n, of basis $(\mathbf{e_1}, \ldots, \mathbf{e_n})$. It is equipped with a norm, called the Euclidean norm $\|\cdot\|_2$:

$$\left\| \sum_{i=1}^{n} \alpha_i \mathbf{e_i} \right\|_2 = \sqrt{\sum_{i=1}^{n} \alpha_i^2}.$$

This norm belongs to a larger family, called the p-norms:

$$\left\| \sum_{i=1}^{n} \alpha_i \mathbf{e_i} \right\|_p = \left(\sum_{i=1}^{n} \alpha_i^p \right)^{1/p}.$$

It is, however, the only norm of this family that is induced by a scalar product, given by

$$\left\langle \sum_{i=1}^{n} \alpha_i \mathbf{e_i}, \sum_{j=1}^{n} \beta_j \mathbf{e_j} \right\rangle = \sum_{i=1}^{n} \alpha_i \beta_i.$$

Theorem 1. *All norms on $\mathbb{R}^n$ are equivalent, i.e. for $\|\cdot\|, |||\cdot|||$ two norms on $\mathbb{R}^n$, there exists $c, C > 0$ such that for all $x \in \mathbb{R}^n$,*

$$c\|x\| \leq |||x||| \leq C\|x\|.$$

2 Sequences

Definition 3 (Convergent sequences). *A sequence* $(x_n)_{n\geq 1}$ *in* $(\mathbb{R}^n, \|\cdot\|)$ *converges to* $x \in \mathbb{R}$ *if for all* $\epsilon > 0$*, there exists* $N \geq 1$ *such that for all* $n \geq N$*,* $\|x_n - x\| \leq \epsilon$*. We denote* $x_n \xrightarrow[n\to\infty]{} x$*.*

Definition 4 (Cauchy sequences). *A sequence* $(x_n)_{n\geq 1}$ *in* $(\mathbb{R}^n, \|\cdot\|$ *is Cauchy if for all* $\epsilon > 0$*, there exists* $N \geq 1$ *such that for all* $n, m \geq N$*,* $\|x_n - x_m\| \leq \epsilon$*.*

Theorem 2 (Completeness of $\mathbb{R}^n$). *A sequence* $(x_n)_{n\geq 1}$ *in* $(\mathbb{R}^n, \|\cdot\|$ *is Cauchy if and only if it converges.*

3 Reminders of topology in $\mathbb{R}^n$

Definition 5 (Open sets). *For all* $x \in \mathbb{R}^n$ *equipped with a norm* $\|\cdot\|$ *and* $r > 0$*, we denote*

$$B(x, r) := \{y \in \mathbb{R}^n : \|x - y\| < r\}.$$

A subset $U \subset \mathbb{R}^n$ *is open if for all* $x \in S$*, there exists* $r > 0$ *such that* $B(x, r) \subset S$*. Notice that* $\emptyset$ *and* $\mathbb{R}^n$ *are open, that an arbitrary union of open sets is open, and that a finite intersection of open sets is open.*

A subset $S \subset \mathbb{R}^n$ *is closed if and only if* S^c *is open. In particular,* $\emptyset$ *and* $\mathbb{R}^n$ *are closed, an arbitrary intersection of closed sets is closed, and a finite union of closed sets is closed.*

Remark 3 (Terminology is unfortunate). *A subset* $U \subset \mathbb{R}^n$ *can be: open, closed, open and closed, neither open nor closed.*

Proposition 4 (Characterization via sequences). *A subset* $S \subset \mathbb{R}^n$ *is closed if and only if for every sequence* $(x_n)_{n\geq 1}$ *that converges, the limit belongs to* S*.*

Theorem 5 (Bolzano Weierstrass – Sequential compactness in $\mathbb{R}^n$). *A subset* $S \subset \mathbb{R}^n$ *is sequentially compact, i.e. with the property that every sequence* $(x_n)_{n\geq 1} \subset S$ *admits a subsequence converging in* S*, if and only if* S *is bounded and closed. See the reminders sheet for a proof.*

4 Continuous functions

Definition 6. *A function* $f : (\mathbb{R}^n, |||\cdot|||) \to (\mathbb{R}^m, \|\cdot\|)$ *is continuous at* $x \in \mathbb{R}^n$ *if for all* $\epsilon > 0$*, there exists* $\delta > 0$ *such that for all* $y \in B(x, \delta)$*,* $\|f(x) - f(y)\| \leq \epsilon$*.*

Equivalently, f *is continuous at* x *if and only if for all* $(x_n)_{n\geq 1} \subset \mathbb{R}^n$ *converging to* x*,* $f(x_n)$ *converges to* $f(x)$ *in* $(\mathbb{R}^m, \|\cdot\|)$*.*

f *is said continuous if it is continuous at every* $x \in \mathbb{R}^n$*.*

Definition 7. *A function* $f : (\mathbb{R}^n, |||\cdot|||) \to (\mathbb{R}^m, \|\cdot\|)$ *is uniformly continuous if for all* $\epsilon > 0$*, there exists* $\delta > 0$ *such that for all* $x, y \in \mathbb{R}^n$ *such that* $|||x - y||| < \delta$*,* $\|f(x) - f(y)\| \leq \epsilon$*.*

Both definitions above can trivially be adapted when f takes value from a subset $E \subset \mathbb{R}^n$.

Proposition 6. *Let $K \subset \mathbb{R}^n$ be non-empty, closed and bounded, and $f : K \to \mathbb{R}^m$ be continuous. Then f is uniformly continuous.*

Proposition 7 (Maximum principle)**.** *Let $K \subset \mathbb{R}^n$ be non-empty, closed and bounded, and $f : K \to \mathbb{R}^m$ continuous. Then there exist $\underline{x}, \overline{x} \in K$ such that*

$$f(\underline{x}) = \inf_{x\in K} f(x), \quad f(\overline{x}) = \sup_{x\in K} f(x).$$

See the reminders sheet for a proof.

Proposition 8 (Intermediate value theorem)**.** *Let $K \subset \mathbb{R}^n$ be non-empty and path-connected, and $f : K \to \mathbb{R}^m$ continuous. Then for all $y \in (\inf_{x\in K} f(x), \sup_{x\in K} f(x))$, there exists $x \in K$ such that $f(x) = y$.*

5 Riemann integral

Definition 8. *Let $a < b \in \mathbb{R}$. A partition of $[a, b]$ is a finite set $P = \{a_0, a_1, \dots, a_n\}$ with*

$$a = a_0 < a_1 < \cdots < a_n = b.$$

For $f : [a, b] \to \mathbb{R}$ bounded, we define the upper and lower Riemann sums

$$U(P, f) = \sum_{i=1}^{n} \left(\sup_{a_{i-1}\le x\le a_i} f(x) \right) (a_i - a_{i-1}), \quad L(P, f) = \sum_{i=1}^{n} \left(\inf_{a_{i-1}\le x\le a_i} f(x) \right) (a_i - a_{i-1})$$

f is said to be Riemann-integrable if

$$\inf_P U(P, f) = \sup_P L(P, f),$$

(where the sup and inf are taken over all partitions of $[a, b]$) and this value is denoted $\int_a^b f$.

Theorem 9. *$f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for every $\epsilon > 0$ there exists a partition P_ϵ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.*

Theorem 10. *Let $a < b \in \mathbb{R}$ and $(f_n)_{n\ge1} \subset C([a, b], \mathbb{R})$ converging uniformly to f. Then*

$$\int_a^b f_n(x)\mathrm{d}x \xrightarrow[n\to\infty]{} \int_a^b f(x)\mathrm{d}x.$$

6 Derivatives

6.1 In $\mathbb{R}$

Definition 9. *Let $a < b \in \mathbb{R}$. We say that $f : (a, b) \to \mathbb{R}$ is differentiable at $x \in (a, b)$ if the following limit*

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

exists, in which case we write it $f'(a)$. We further write $f \in C^1((a, b), \mathbb{R})$ if f is differentiable at every point of (a, b) and $f' : x \mapsto f'(x)$ is continuous on (a, b).

The spaces $C^k((a, b), \mathbb{R})$ are defined analogously.

Theorem 11 (Mean value theorem)**.** *Let $f \in C^0([a, b], \mathbb{R})$ be differentiable on (a, b). There exists $c \in (a, b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Theorem 12 (Fundamental theorem of calculus)**.** *Let $f \in C^0([a, b], \mathbb{R})$ and consider*

$$F : x \mapsto \int_a^x f(y)\mathrm{d}y, \quad x \in [a, b].$$

Then $F \in C^1((a, b), \mathbb{R})$ and for all $x \in (a, b)$, $F'(x) = f(x)$.

Note that the converse of this theorem is not true, in that differentiability of F need not imply continuity of f.

6.2 In $\mathbb{R}^n$

Definition 10. *Let $U \subset \mathbb{R}^n$ be open. A function $f : U \to \mathbb{R}^m$ is said to be differentiable at $x_0 \in U$ if there exists a linear map $Df(x_0) \in \mathbb{R}^{m \times n}$ such that for all $x \in \mathbb{R}^n$,*

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + o(\|x - x_0\|).$$

Theorem 13. *Let $U \subset \mathbb{R}^n$ be open. If $f : U \to \mathbb{R}^m$ is differentiable at $x_0 \in U$, then $Df(x_0)$ is unique and is given by*

$$(Df(x_0))_{ij} = \lim_{t \to 0} \frac{f_i(x_0 + te_j) - f_i(x_0)}{t} =: \frac{\partial f_i}{\partial x_j}(x_0)$$

Furthermore, f is continuous at x_0.

We write that $f \in C^1(U, \mathbb{R}^m)$ if f is differentiable at every $x \in U$ and the function $Df : U \to \mathbb{R}^{m \times n}, Df : x \mapsto Df(x)$ is continuous. Inductively, we say that $f \in C^k(U, \mathbb{R}^m)$ if for $0 \leq i \leq k-1$, $D^{(i)}f$ is differentiable with differential $D^{(i+1)}f$, with the latter continuous.

Proposition 14. *Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$ and $x_0 \in U$ be such that there exists $\delta > 0$ such that for all $1 \leq i \leq n, 1 \leq j \leq m$, $\frac{\partial f_i}{\partial x_j}(x)$ exists for all $x \in B(x_0, \delta)$ and is continuous at x_0. Then f is differentiable at x_0.*

Recall that the condition in the proposition is sufficient for differentiability, but not necessary.

7 Miscellaneous

Theorem 15 (Banach fixed point theorem)**.** *Let (X, d) be a complete metric space, $f : X \to X$ and $0 \leq L < 1$ such that for all $x, y \in X$,*

$$d(f(x), f(y)) < Ld(x, y).$$

Then f admits a unique fixed point, i.e. there exists a unique $x \in X$ such that $f(x) = x$.

Note that the theorem does not hold for $L = 1$: consider for example $X = \mathbb{R}$ and $f : x \mapsto \pi/2 + \arctan(x)$.