Exercise sheet 1

Counterexamples

Often in solving exercises it's important to build a simple mental picture.

Exercise 1 (A non-exercise)**.** *Can you explain with a drawing what it means for a function to be continuous? Uniformly continuous? For a sequence to converge? To be Cauchy?*

It also helps when finding examples and counterexamples.

Exercise 2. *Find a sequence* $(f_n)_{n\geq 1}$ *of continuous functions on* $[0, 1]$ *that converges pointwise to a function which is not continuous*

Often counterexamples are built on a simple observation. Here: do you remember what are the key examples of non-Riemann-integrable functions?

Exercise 3. *Find sequences* $(f_n)_{n\geq 1} \subset \mathbb{R}^{[0,1]}$ *of Riemann-integrable functions converging pointwise to* $f : [0, 1] \to \mathbb{R}$ *such that*

1. *(Limits of Riemann-integrable functions need not be Riemann-integrable)* f *is not Riemann integrable.*
2. *(Limits of integrals doesn't equal the integral of the limit)* f *is Riemann integrable, but* $\int_0^1 f_n(x)\mathrm{d}x$ *does not converge to* $\int_0^1 f(x)\mathrm{d}x$*.*

How to prove something?

Let us work through how one would try to prove a theorem, based on a theorem you have already seen.

Exercise 4 (Sequences, I)**.** *We work towards proving the Bolzano-Weierstrass theorem: for any* $N \geq 1$ *and* $K \subset \mathbb{R}^N$ *closed and bounded, every sequence* $(x_n)_{n\geq 1} \subset K$ *admits a subsequence* (x_{n_k}) *that converges in* K*.*

Tips. *There are some natural first steps when facing a statement to prove:*

- *Simplify: Here one can start for example with the simpler-looking case* $N = 1$*.*
- *Simplify more: What is the simplest closed and bounded set in* $\mathbb{R}$*? Can you prove it in this case?*
- *Check conditions: Why do we ask closedness? Why do we ask boundedness?*

Steps.

1. *Does the statement hold if* K *was equal to* $(0, +\infty)$ *or* $(0, 1)$*?*
2. *Now consider the case* $K = [0, 1]$*. Which criteria do you know for proving a convergence of a sequence? Each of the following gives a proof*
   - *Cauchy: Show that for each* $n \geq 1$*, there must exist* $i \in \{0, \dots, 2^n - 1\}$ *such that* $[i2^{-n}, (i + 1)2^{-n}]$ *contains infinitely many elements of* $(x_n)_{n\geq 1}$ *and conclude.*
   - *Monotonicity: Show that the sequence* $y_n = \inf_{k\geq n} x_k, n \geq 1$*, is increasing and conclude*

3. *Where do these proof fail in the cases $K = (0, \infty)$ or $K = (0, 1)$?*

4. *Can you now generalize to K any closed*[1] *and bounded set?*

5. *What happens now if we work with $K \subset \mathbb{R}^N$ closed and bounded?*

Exercise 5 (Sequences, II)**.** *We now work towards proving the following theorem: let $K \subset \mathbb{R}^N$ be closed and bounded and $f \in C(K, \mathbb{R})$. Prove that there exist $\underline{x}, \overline{x} \in K$ such that*

$$f(\underline{x}) = \inf_{x \in K} f(x), \quad f(\overline{x}) = \sup_{x \in K} f(x).$$

We encourage you to decompose the problem as in the previous exercise in order to work towards a solution. If you are stuck, we offer a roadmap at the end of the sheet.

How to define something?

Mastering infinite sums will be of use in the course, as we often work with series of functions.

Exercise 6. *In this exercise, we try to generalise the definition of sums that you have already seen:*

1. *Recall the definition of a converging and absolutely converging sequence, and the relevant results.*

2. *Consider a convergent series $\sum_{n=1}^{+\infty} a_n$, i.e. such that $\lim_{N \to \infty} \sum_{n=1}^{N} a_n$ exists. Is it true that*

$$\sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{+\infty} a_{2n} + \sum_{n=1}^{+\infty} a_{2n-1},$$

   i.e. can you sum the even and odd terms separately? (this writing implicitly requires to check if the series converge) If not, what condition can you add to make it true?

3. *Suppose now that for each $n \in \mathbb{Z}$ we have some number $a_n \in \mathbb{R}$, and we want to define something like*

$$\sum_{n \in \mathbb{Z}} a_n.$$

   How would you make sense of this? And if you can think of different ways to define it, can you find sufficient conditions so that they agree?

4. *In particular, is it true that if $\sum_{n \in \mathbb{Z}} a_n$ converges, then $\sum_{n=1}^{+\infty} a_n$ (and $\sum_{n=1}^{+\infty} a_{-n}$) does too, and*

$$\sum_{n \in \mathbb{Z}} a_n = a_0 + \sum_{n=1}^{+\infty} a_n + \sum_{n=1}^{+\infty} a_{-n}.$$

Tips. *It is good to have a bag of sequences in mind: for instance, $a_n = \operatorname{sign}(n), a_n = (-1)^n, a_n = 1/n$... etc...*

[1] Here we mean that K is closed if for all $x \in \mathbb{R} \setminus K$, there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset \mathbb{R} \setminus K$.

Steps (Exercise 5).

- *The first question is the same: what would it mean for the statement to be wrong? Then the first recommendations are as before: consider first* $N = 1$, $K = [a, b]$, *and identify why the result would not be true for closed and bounded intervals, as well as non-continuous functions.*
- *When still unsure what to do, one can always try to rewrite the quantities at play with equivalent definitions. In particular, how can you relate sequences and infima/suprema? Can you then use what you proved in the previous exercise?*
- *Conclude in the case* $N = 1, K = [a, b]$, *and see if your proof extends to the general case!*