ANALYSIS IV FOR PHYSICS

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Section 0

Introduction and motivation

As motivation, let us consider the mathematical description of heat transmission on a homogeneous circular rod: the heat equation.

The heat equation on an interval $[0, 1)$ (describing the rod) is given by describing the evolution of the temperature profile

$$\frac{\partial u(t,x)}{\partial t} = D\Delta u(t,x)$$

together with some initial condition $u_0(x) = u(0,x)$ and the boundary condition $u(t,0) = u(t,1)$ for all $t \geq 0$ to express that the ends of the rod are connected. Recall that in the 1D case $\Delta f := \frac{\partial^2 f}{\partial x^2}$ and $D > 0$ is the diffusion coefficient.

The revolutionary idea of Fourier was as follows. He noticed empirically that the heat profile over time shows spatially oscillatory behaviour, and thus also motivated by the solution of the wave equation using waves, he proposed to write any solution using spatially oscillating functions like $f_n(x) = \sin(2\pi x)$ and $g_n(x) = \cos(2\pi nx)$. More precisely, one could try to find a solution of the form

$$u(t,x) = \sum_{n\geq 1} s_n(t)\sin(2\pi nx) + \sum_{n\geq 0} c_n(t)\cos(2\pi nx).$$

But now notice that $\Delta f_n = -4D\pi^2 n^2 f_n$ and thus if we try a solution of the form $u_n(x,t) = f_n(x)s_n(t)$ with f_n as above, we obtain an equation

$$\frac{\partial s_n(t)}{\partial t} = -4D\pi^2 n^2 s_n(t).$$

This is a well-known ODE that is easily solved: $s_n(t) = \exp(-4D\pi^2 n^2 t)s_n(0)$. Similarly for the cos terms we get $c_n(t) = \exp(-4D\pi^2 n^2 t)c_n(0)$.

We conclude that it would make sense to propose a solution of the form

$$u(t,x) = \sum_{n\geq 1} s_n(0)\exp(-D4\pi n^2 t)\sin(2\pi nx) + \sum_{n\geq 0} c_n(0)\exp(-D4\pi n^2 t)\cos(2\pi nx).$$

Notice that the initial condition then translates to the condition:

$$u_0 = \sum_{n\geq 1} s_n(0)\sin(2\pi nx) + \sum_{n\geq 0} c_n(0)\cos(2\pi nx).$$

If we do find such $(s_n(0), c_n(0))_{n\geq 0}$, then we may have found at least one solution to the heat equation on the circular rod.

Now, this may sound very convincing, but on a closer look there are several questions here:

(1) We have infinite sums - do they even converge? When do they converge and in which sense?
(2) For which functions u_0 does the above-given expansion hold? In other words for which initial conditions can we find a solution by this method?
(3) Are such expansions unique? Are the solutions we find unique?

(4) Can one approximate solutions? For example this is relevant when trying to numerically solve the equation. This is a question about convergence - and further, how does the notion of convergence relate to the coefficients s_n, c_n?

(5) More generally, how should one measure closesness of different initial conditions, different solutions?

(6) What happens for non-circular rods, e.g. rods with endpoints in heat-baths? Or in higher dimensions?

(7) What about more non-homogeneous case where D is no longer a constant in space? Or when we replace Δ with more general (linear) operators, including for example also some outside influences?

The aim of this course is to study the right mathematical framework to ask and answer such questions. This will bring us to study function spaces, the Lebesgue integral and spectral theory of linear operators. To see why some of those aspects might enter let us further consider a simplified model.

0.1 A discrete model

To understand what we may hope to achieve, let us consider the same problem of heat diffusion but on a discretised space. For example we think that the rod instead is decomposed of n small containers which can exchange heat between its neighbours.

The temperature profile is now given by $u(x,t) : \{0, 1, \dots, n\} \to \mathbb{R}$, with the periodicity condition $u(0,t) = u(n,t)$ for all $t \geq 0$.

The evolution is still given by

$$\frac{\partial u(t,x)}{\partial t} = K\Delta_d u(t,x)$$

together with some initial condition $u_0(x) = u(0,x)$, only instead of the real Laplacian, we have the discrete Laplacian $\Delta_d f(x) := \frac{1}{d_x}\sum_{y\sim x} f(y) - f(x)$, where $y \sim x$ means that y, x are neighbours in the underlying discrete graph and d_x is the number of neighbours of the vertex x. In our concrete case we have a circular graph with n vertices and thus $\Delta_d f(x) := \frac{f(y)+f(z)-2f(x)}{2}$, where y, z denote the neighbouring vertices.

Now notice that now the problem is really a system of n coupled ordinary differential equations of second degree and Δ_d is just a linear operator on $\mathbb{R}^n \to \mathbb{R}^n$. So how do we solve it?

Let use the same steps as above but see that they have a very simple and concrete meaning here:

- Notice that each u_t can be seen as a vector in $\mathbb{R}^n$ with coordinates and Δ_d can be seen as a symmetric linear operator on $\mathbb{R}^n$(check it!)
- As such Δ_d can be diagonalized: there is an orthonormal basis $\phi_1, \dots, \phi_n$ and eigenvalues $\lambda_1, \dots, \lambda_n$ such that $\Delta_d \phi_i = \lambda_i \phi_i$. In particular any function $u : \mathbb{R}^n \to \mathbb{R}^n$ can be uniqueley written as $\sum_{i=1}^n c_i \phi_i$.
- But now if we write $u_i(t) := c_i(t)\phi_i$, then again each $c_i(t)$ satisfies now a decoupled ODE
$$\frac{\partial c_i(t)}{\partial t} = K\lambda_i s_n(t)$$
and thus has a solution $c_i(0)\exp(K\lambda_i t)$.

- We conclude a solution by finding $c_i(0)$ by determining the unique expansion $u_0 := \sum_{i=1}^n c_i(0)\phi_i$.
- Given the uniqueness of the expansion, this solution is also unique.
- And finally, we can easily compare solutions just using for example the Euclidean norm. For example conclude that if the initial conditions are close, then so will be the solutions at all times $t > 0$. We also know that this distance is equivalently measured using the distances between two sets of coefficients $(c_i)_{i=1\dots n}, (\tilde{c}_i)_{i=1\dots n}$ - and here using the Euclidean norm instead of some other norm is important.

Hence in this set-up all works super well and would work equally well as long as we have a symmetric linear operator L instead of Δ_d.

What did we use here?

- We used the fact that $\mathbb{R}^n$ is finite-dimensional and thus there exist basis that give unique expansions for each vector
- We used the fact that Δ_d is linear and symmetric and by the spectral theorem can be diagonalised and we can find a basis of eigenvectors
- We used implicitly the linearity of the equation

None of these facts are clear in our original set-up as the space of functions from $[0, 1]$ to $\mathbb{R}$ is no longer finite-dimensional!

To address those we will have to look at spaces of functions and try to first see which such spaces have a nice structure. For example, which spaces of functions satisfy linearity? Which can be define a norm and talk about orthonormality? For which spaces do we have orthonormal expansions? Looking for such nice properties brings us for example to also introduce the Lebesgue integral to construct nice basis of functions.

After that, having spent some time understanding function spaces, we briefly at the study of linear operators on such spaces and in particular find some set-ups where there are similar orthonormal decompositions using eigenfunctions. We then put all this together to rigorously explain solving the inhomogeneous heat equation and other similar problems.

But this is already enough of introduction, let us get going!

Section 1

The space of continuous functions

Let us start with maybe the most intuitive of function spaces - the space of continuous functions. This is partly a recap, as you have been working with continuous functions in Analysis I-III, and we are just putting things in a wider context.

To start off the functions will be taking values on closed boxes $D \subseteq \mathbb{R}^n$, i.e. rectangles $[a_1, b_1] \times \cdots \times [a_n, b_n]$) and taking values in $\mathbb{R}$. At the end of the section we will discuss to what extent we can (and may want to) generalize both of these choices. You may safely just suppose $D = [0,1]$, as no actual extra difficulty comes from going to higher dimensions.

The set of all continuous functions from $D \to R$ will be denoted by $C(D, \mathbb{R})$:

$$C(D, \mathbb{R}) := \{f : D \to R, f \text{ continuous}\}.$$

In what follows we will try to understand the structure of this space.

1.1 Vector space structure of $C(D, \mathbb{R})$

The first observation we can make about the space $C(D, R)$ is that it has a linear structure like for example the vector space $(\mathbb{R}^n, +)$: if $f, g \in C(D, \mathbb{R})$, then also the function $h(x) := f(x) + g(x)$ is in $C(D, \mathbb{R})$, as is $\lambda f(x)$ where $\lambda \in \mathbb{R}$.

Let us quickly check this for the first statement: for every $x \in D$, by continuity of f, g we can choose δ_f, δ_g such that if $y \in D, \|x - y\| < \delta_f$ then $|f(x) - f(y)| \leq \frac{\epsilon}{2}$ and if $y \in D, \|x - y\| < \delta_g$, then $|g(x) - g(y)| \leq \frac{\epsilon}{2}$. But this means that if $\|x - y\| < \min(\delta_f, \delta_g)$, we have that $|h(x) - h(y)| < \epsilon$ by the triangle inequality.

Exercise 1.1. *Show that in fact $C(D, \mathbb{R})$ has also multiplicative structure: i.e. if $f, g \in C(D, \mathbb{R})$, then also the product $h(x) := f(x)g(x)$ is in $C(D, \mathbb{R})$. What about the function $\max(f, g)$?*

In fact, the space $C(D, \mathbb{R})$ with addition satisfies all the axioms of a vector space! Indeed, the identity element would be just the constant zero function, the inverse element of f the function $-f$ and all conditions are nicely met, as you can easily and patiently check.

Exercise 1.2. *Recall the axioms of a vector space and verify them in the case of $(C(D, \mathbb{R}), +)$.*

In what follows we will often also call the vector space just $C(D, \mathbb{R})$.

Now we might be also interested in summing infinitely many functions, i.e. looking at sums $\sum_{n \geq 1} f_n$. But in what sense can we talk about it? More generally, given a sequence of $(g_n)_{n \geq 1}$ in which sense can we talk about its convergence and limit?

The first idea might be to define limits pointwise: for each $x \in D$ the sequence $(g_n(x))_{n \geq 1}$ is just a sequence of real numbers and thus we know what its convergence means. Thus we may want to define the convergence of $(g_n)_{n \geq 1}$ as functions to mean the convergence of $(g_n(x))_{n \geq 1}$ for all $x \in D$. This is called pointwise convergence and as you have already seen it suffers a small drawback:

Exercise 1.3. *For D a closed box in $\mathbb{R}^n$, find a sequence of functions in $C(D, \mathbb{R})$ that converges pointwise to a function that is not continuous.*

It is a good idea to start from the case $D = [0, 1]$ (which we discussed in class), but then think how to do it in general.

We will look for other notions of convergence and to do this will introduce a norm on the set $C(D, \mathbb{R})$.

1.2 The uniform norm on $C(D, \mathbb{R})$

Recall that the vector space $\mathbb{R}^n$ comes also with several natural norms that give a notion of length of a vector and give us a way to measure distances between vectors. It comes out that one can also endow $C(D, \mathbb{R})$ with a natural norm.

Definition 1.1 (The supremum (or uniform) norm)**.** *For $f \in C(D, \mathbb{R})$ we define its supremum (or uniform) norm $\|f\|_\infty := \sup_{x \in D} |f(x)|$.*

In this definition we use the fact that D is closed and bounded - otherwise the supremum might not be finite.

Exercise 1.4. *Find an example of D that is not closed or not bounded, and $f \in C(D, \mathbb{R})$ such that $\|f\|_\infty$ as defined above is infinite.*

We called the expression above a norm, but recall that a norm on a vector space has again a precise mathematical definition and its conditions need to be checked:

Proposition 1.2. *$\|f\|_\infty$ indeed defines a norm on the vector space $C(D, \mathbb{R})$.*

Proof. We need to check the conditions for a norm.

(1) $\|f\|_\infty \geq 0$ with equality if and only if f is equal to the constant zero function. This is clear.
(2) $\|\lambda f\|_\infty = |\lambda| \|f\|_\infty$ is also clear.
(3) Finally, we need to check the triangle inequality $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$. We have

$$\|f + g\|_\infty = \sup_{x \in D} |f(x) + g(x)| \leq \sup_{x \in D} (|f(x)| + |g(x)|)$$

by the triangle inequality. But now

$$\sup_{x \in D} (|f(x)| + |g(x)|) \leq \sup_{x \in D} |f(x)| + \sup_{x \in D} |g(x)|$$

and we conclude.

□

Thus $(C(D, \mathbb{R}), +, \| \cdot \|_\infty)$ is a normed vector space pretty much like $\mathbb{R}^n$ with any of these norms. This gives us a way to talk about convergence that is much more natural:

Proposition 1.3. *Let $(f_n)_{n \geq 1}$ be a sequence of $C(D, \mathbb{R})$ functions converging to some function $f : D \to \mathbb{R}$ w.r.t. the uniform norm. Then in fact f is continuous.*

This is a restatement of a result from Analysis I which says that pointwise limits of continuous functions are not continuous.

The proof technique is called the 3ϵ or $\epsilon/3$ argument and you have again seen it already in Analysis I. Let us give the proof just to understand what is now different from the earlier situation

Proof. It suffices to show that for every $x \in D$, we can find $\delta > 0$ such that $|f(x) - f(y)| < 3\epsilon$ whenever $\|x - y\| < \delta$.

We can first choose a fixed $n \in \mathbb{N}$ large enough so that $\|f_n - f\|_\infty < \epsilon$, and in particular $|f_n(x) - f(x)| < \epsilon$ for every $x \in D$ by the definition (these are the first two epsilons).

Further, by continuity of f_n we can choose $\delta > 0$ such that for every $y \in D$ with $\|x-y\| < \delta$, we have that $|f_n(x) - f_n(y)| < \epsilon$ (this is the third epsilon). Putting things together using triangle inequality we obtain:

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_n(x) + f_n(x) - f_n(y) + f_n(y) - f(y)| \leq \\ &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < 3\epsilon. \end{aligned} \tag{1.1}$$

□

Notice that for pointwise convergence the first step fails: we can not necessarily choose an n such that $\sup_{x \in D} |f_n(x) - f(x)| < 3\epsilon$.

Thus using this norm the set $C(D, \mathbb{R})$ is also closed under taking convergent sequences. In fact, it is even nicer than that and there are no gaps at all in the space, e.g. the space is complete - a notion you have met for $\mathbb{R}^n$ and that we recall here.

Definition 1.4 (Completeness of a normed space). *A normed space $(X, \|\cdot\|)$ is called complete if every Cauchy sequence $(x_n)_{n\geq 1}$ (i.e. every sequence such that for every $\epsilon > 0$, there is an n_ϵ with $\|x_n - x_m\| \leq \epsilon$ for all $n, m \geq n_\epsilon$) converges to an element $x \in X$.*

Theorem 1.5. *The space $(C(D, \mathbb{R}), +, \|\cdot\|_\infty)$ is a complete normed vector space.*

The idea is to us completeness of $\mathbb{R}$ to define a potential limiting function and then to verify that it really is that function.

Proof. We only need to check the completeness. So let $(f_n)_{n\geq 1}$ be a Cauchy sequence in $C(D, \mathbb{R})$. As for every $x \in D$, $(f_n(x))_{n\geq 1}$ is Cauchy and $\mathbb{R}$ is complete, we now a limit exists and we can denote this limit by $f(x)$. It remains to see that $f_n \to f$ in the uniform norm and that f is continuous. The latter claim follows from the proposition above, so we need to just prove the convergence w.r.t. the uniform norm. This is left as an exercise on the exercise sheet.

□

Remark 1.6. *Mathematicians call any normed vector space that is complete a Banach space. Such spaces are quite important in setting up quantum field theory.*

The completeness of the space has important application, one of them is finding solutions to ODEs via approximation. The tool used there is the Banach contraction mapping theorem that you have already met in Analysis II according to the course sheets and that is just recalled here:

Theorem 1.7 (Banach contraction mapping theorem). *Let $F : C(D, \mathbb{R}) \to C(D, \mathbb{R})$ be contractive w.r.t. the uniform norm: $\|F(f) - F(g)\|_\infty < C\|f - g\|_\infty$ with $C < 1$. Then there is a unique solution to $F(f) = f$ that can be obtained from the limit $\lim_{n\to\infty} F^{(n)}(f)$.*

1.3 Fourier series for continuous functions

The expansion of a function f on $[0,1]$ to a series of the form

$$f(x) = \sum_{n\geq 1} s_n \sin(2\pi n x) + \sum_{n \geq 0} c_n \cos(2\pi n x) \tag{1.2}$$

is called the Fourier expansion or Fourier series. We saw in the introduction that it could be quite useful, but we didn't see any results on its existence / uniqueness. So let us look at this in the context of continuous functions f now.

In fact we will see that these questions resolve themselves very smoothly once we find the "right functional space", but it is instructive to consider the questions already.

The first question is how should we go about finding the coefficients s_n, c_n? There the key is the following lemma.

Lemma 1.8. *The following orthogonality relations hold for integers $m, n \geq 0$:*

1. *Cosine-cosine Orthogonality:*

$$\int_0^1 \cos(2\pi n x)\cos(2\pi m x)\,dx = \begin{cases} 1, & \text{if } n = m = 0, \\ \frac{1}{2}, & \text{if } n = m \neq 0, \\ 0, & \text{if } n \neq m. \end{cases}$$

2. *Sine-Sine Orthogonality:*

$$\int_0^1 \sin(2\pi n x)\sin(2\pi m x)\,dx = \begin{cases} 0, & \text{if } n = 0 \text{ or } m = 0, \\ \frac{1}{2}, & \text{if } n = m \neq 0, \\ 0, & \text{if } n \neq m. \end{cases}$$

3. *Sine-Cosine Orthogonality:*

$$\int_0^1 \sin(2\pi n x)\cos(2\pi m x)\,dx = 0 \quad \forall n, m.$$

Proof. The proof is a simple consequence of trigonometric identities and their integrals and is left for the exercise sheet. □

The consequence of this observation is that if we expect the representation (1.2) to hold in any nice sense, then the coefficients s_n, c_n should be given by:

- Cosine Coefficients c_n:

$$c_n = 2\int_0^1 f(x)\cos(2\pi n x)\,dx, \quad \text{for } n \geq 1.$$

  For the constant term c_0, we have:

$$c_0 = \int_0^1 f(x)\,dx.$$

- Sine Coefficients s_n:

$$s_n = 2\int_0^1 f(x)\sin(2\pi n x)\,dx, \quad \text{for } n \geq 1.$$

Further notice that if we want it to hold at the endpoints, then we better have $f(0) = f(1)$ as this also holds for every function in the series.

Maybe a bit surprisingly both the existence and uniqueness are really not clear even for continuous functions!

Indeed, the understanding of counterexamples has evolved with time. The first observation is as follows

- There exists a continuous function f satisfying $f(0) = f(1)$ whose Fourier series converges pointwise everywhere but does not converge uniformly.

It is not easy to come up with such a function but once given, it is easy to check (probably on the exercise sheet).

A more stunning claim comes from the second half of 19th century from Du Bois-Reymond:

- There exist continuous functions $f \in C([0,1],\mathbb{R})$ with $f(0) = f(1)$ such that the Fourier series diverges at a point $x \in [0,1]$.

This was then extended by several people to find continuous functions $f \in C([0,1],\mathbb{R})$ with $f(0) = f(1)$ where the Fourier series diverges at infinitely many points. Further, in the beginning of 20th century Kolmogorov showed that there are functions for which this divergence happens at almost all points in a very precise sense that we meet later. Finally, Katznelson showed in 1970s that in fact for every continuous function f and every $\epsilon > 0$, there is some continuous function g with $\|g - f\| < \epsilon$ and the Fourier series of g diverges at some point meaning that these unpleasant functions are really everywhere!

Yet, the situation is not completely hopeless either:

Proposition 1.9. *Let $f \in C^2([0,1])$ be twice continuously differentiable and satisfying $f(0) = f(1)$ and $f'(0) = f'(1)$. Then its Fourier series*

$$f(x) = \lim_{N\to\infty} \sum_{n\le N} (s_n \sin(2\pi nx) + c_n \cos(2\pi nx),$$

converges w.r.t. $\|\cdot\|_\infty$.

Remark 1.10. *In fact the result holds under much less stringent conditions, e.g. when the functions are so-called Holder continuous, i.e. satisfying $|f(x) - f(y)| < |x-y|^a$ for some $a > 0$. Just the proof then needs a bit more care and is out of the scope for us.*

The key ingredient is the following lemma, which we observed when guessing the solution to the heat equation and that really explains why Fourier series are so useful:

Lemma 1.11. *Suppose that $f \in C([0,1],\mathbb{R})$ is k times continuously differentiable and satisfies $f^j(0) = f^j(1)$ for all $j = 0 \dots k-1$ [1] Then there is some $C > 0$ such that for all $n \ge 1$ $|c_n| \le Cn^{-k}$ and $|s_n| \le Cn^{-k}$.*

The full proof is on the exercise sheet, but let's see the case $k = 1$.

- We have by integration by parts that

$$\int_{[0,1]} \sin(2\pi nx) f(x) dx = \frac{1}{2\pi n} \int_{[0,1]} \cos(2\pi nx) f'(x) \le \frac{1}{2\pi n} \|f'\|_\infty.$$

[1]Here by $f^j(x)$ we mean the $j-th$ derivative of f at x, the $0-$th derivative being the function itself.