Exercise sheet 7

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

Reminders

Exercise 1 (f^{-1} is very nice)**.** *Let $f : A \to B$, I_A, I_B some sets of indices, and $(B_i)_{i\in I_B}$ any collection of subsets of B. Recall that the preimage $f^{-1}(B_i)$ is defined as*

$$f^{-1}(B_i) = \{x \in A : f(x) \in B_i\}.$$

Check that when f is bijective, f^{-1} is actually a function. What if f is not injective or not surjective?

Then prove the following identities:

- $f^{-1}(\cup_{i\in I_B} B_i) = \cup_{i\in I_B} f^{-1}(B_i)$;
- $f^{-1}(\cap_{i\in I_B} B_i) = \cap_{i\in I_B} f^{-1}(B_i)$.

Proof. If f is bijective, for any $y \in B$, by surjectivity there exists $x \in A$ such that $f(x) = y$, and by injectivity any $x' \in A$ satisfying $f(x') = y$ must be such that $x' = x$. We can therefore define $f^{-1}(y) := x$, so that f^{-1} is a well-defined function $B \to A$. When f is not injective or surjective, f^{-1} is not a well-defined function (consider for example $x \mapsto x^2$ from $[-1, +1]$ to $[0, +1]$ not injective, $x \mapsto x^2$ from $[0, +1]$ to $[-1, +1]$ not surjective), and the preimage is only defined as a mapping from sets to sets (of potentially different cardinality).

The identities then follow directly from the definition of the preimage:

$$x \in f^{-1}\Big(\bigcup_{i\in I} B_i\Big) \iff f(x) \in \bigcup_{i\in I} B_i \iff \exists\, i \in I : f(x) \in B_i \iff \exists\, i \in I : x \in f^{-1}(B_i).$$

$$x \in f^{-1}\Big(\bigcap_{i\in I} B_i\Big) \iff f(x) \in \bigcap_{i\in I} B_i \iff \forall\, i \in I : f(x) \in B_i \iff \forall\, i \in I : x \in f^{-1}(B_i).$$

□

Measurable functions

Exercise 2 (Other definitions of measurability)**.** *Prove that a function f is measurable if and only if for all $a < b \in \mathbb{R}$, $f^{-1}((a, b))$ is Borel-measurable. Also prove that the same works if one replaces (a, b) with $[a, b]$.*[1]

Proof. The proof of these statements boils down to the following simple equalities: for all $a, b \in \mathbb{R}$ with $a < b$, we have

$$[a, b) = \bigcap_{n\in\mathbb{N}} (a - 1/n, b)$$

and

$$(a, b) = \bigcup_{n\in\mathbb{N}} [a + 1/n, b).$$

[1] Actually, replacing $(a, b), a, b \in \mathbb{R}$ by $(-\infty, b), b \in \mathbb{R}$, or $(a, +\infty), a \in \mathbb{R}$, or $(a, b], a < b \in \mathbb{R}$, etc... lead to all equivalent definitions.

For instance, suppose that $f^{-1}((a,b))$ is measurable for all $a,b\in\mathbb{R}$. Then,

$$f^{-1}([a,b)) = \bigcap_{n\in\mathbb{N}} f^{-1}((a-1/n,b))$$

is also measurable, as the right-hand side is a countable intersection of measurable set. Reciprocally, if $f^{-1}([a,b))$ is measurable for all $a,b\in\mathbb{R}$, then

$$f^{-1}((a,b)) = \bigcup_{n\in\mathbb{N}} f^{-1}([a+1/n,b))$$

is again measurable. The other equivalences are shown under the same principle, using that

$$[a,b] = \bigcap_{n\in\mathbb{N}} [a,b+1/n) = \bigcap_{n\in\mathbb{N}} (a-1/n,b+1/n), \qquad (a,b) = \bigcup_{n\in\mathbb{N}} [a+1/n,b-1/n],$$

et caetera.

Similarly, the claim in the footnote follows from the fact that

$$(-\infty,a) = \bigcup_{n\in\mathbb{N}} (-n,a), \qquad (b,a) = (-\infty,a)\setminus \bigcap_{n=1}^{\infty} (-\infty,\ b+\tfrac{1}{n})$$

□

Exercise 3. *Show that if f,g are measurable, then so are $f+g$ and fg.*

Proof. We have seen two different definitions of measurability: h is measurable iff it is the pointwise limit of simple functions, and if and only if $h^{-1}([a,b))$ is a Borel set for all $a,b\in\mathbb{R}$. We present a solution for each definition.

For the first proof, assume that $f,g:\mathbb{R}^n\to\mathbb{R}$ are measurable and let $f_n,g_n:\mathbb{R}^n\to\mathbb{R}$ be simple functions converging pointwise to f and g respectively, admitting the following expression

$$f_n = \sum_{i=1}^{s_n} a_i^n \mathbf{1}_{F_i^n},$$
$$g_n = \sum_{j=1}^{t_n} b_j^n \mathbf{1}_{G_j^n},$$

with $(a_i^n)_{i=1}^{s_n}, (b_j^n)_{j=1}^{t_n} \subset \mathbb{R}$ and $(F_i^n)_{i=1}^{s_n}$ (resp. $(G_j^n)_{j=1}^{t_n}$) forming a disjoint partition of $\mathbb{R}^n$ by Borel sets. Using that

$$\sum_{i=1}^{s_n} \mathbf{1}_{F_i^n} \equiv \sum_{j=1}^{t_n} \mathbf{1}_{G_j^n} \equiv 1$$

and $\mathbf{1}_{F_i^n}\cdot\mathbf{1}_{G_j^n} = \mathbf{1}_{F_i^n\cap G_j^n}$, we find that

$$f_n+g_n = \sum_{i=1}^{s_n}\sum_{j=1}^{t_n} (a_i^n+b_j^n)\mathbf{1}_{F_i^n\cap G_j^n}$$
$$f_n\cdot g_n = \sum_{i=1}^{s_n}\sum_{j=1}^{t_n} (a_i^n b_j^n)\mathbf{1}_{F_i^n\cap G_j^n},$$

which are simple functions since the intersection of Borel sets is Borel. Since additionally $(f_ng_n)_{n\geq1}, (f_n+g_n)_{n\geq1}$ converge pointwise to fg and $f+g$ respectively, we find that the latter functions are measurable.

We now go over the other proof: in virtue of Exercise 2, we check that $(f+g)^{-1}((a,b)), (fg)^{-1}((a,b))$ are measurable for all $a, b \in \mathbb{R}$. For $f+g$ first, note that $a < f(x)+g(x) < b \Leftrightarrow a - g(x) < f(x) < b - g(x)$ and this holds if and only if there exist $q, r \in \mathbb{Q}$ with $a - g(x) < q < f(x) < r < b - g(x)$, i.e. $q < f(x) < r$ and $a - q < g(x) < b - r$. Hence, we can write

$$(f+g)^{-1}((a,b)) = \bigcup_{q,r\in\mathbb{Q}} f^{-1}((q,r)) \cap g^{-1}((a-q, b-r)),$$

which is a countable union of intersections of measurable sets and hence measurable.

For fg let us first consider the case where $f \geq 0, g \geq 0$. If $b < 0$ we have $(fg)^{-1}((a,b)) = \emptyset$ which is measurable. If $b \geq 0$, denote a^+ the maximum of a and 0. We have similar to before $a^+ < f(x)g(x) < b$ if and only if there exist $q, r \in \mathbb{Q}$ such that $q < f(x) < r$ and $a^+/q < g(x) < b/r$. Hence we have

$$(fg)^{-1}((a,b)) = (fg)^{-1}((a^+,b)) = \bigcup_{q,r\in\mathbb{Q}} f^{-1}((q,r)) \cap g^{-1}((a^+/q, b/r)),$$

which is a countable union of intersections of measurable sets and hence measurable.

For the general case, we can write $f = f^+ - f^-$ where $f^+ = \max\{f, 0\}$ and $f^- = -\min\{f, 0\}$. Similar to above, one can check all cases for $a, b \in \mathbb{R}$ in order to see that $\max\{f,0\}^{-1}((a,b))$ is a measurable set, showing that f^+ is measurable and in a similar way we can show that f^- is measurable. It is straightforward to show that if a function h is measurable, then so is $-h$. We have $fg = f^+g^+ + f^-g^- - f^+g^- - g^+f^-$ which is a sum of measurable functions and hence measurable by the first part of the exercise.

□

Exercise 4. *Show that continuous functions are measurable.*

Proof. Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous: let us show that the sequence of approximations

$$f_n = \sum_{i\in\mathbb{Z}} f(i2^{-n}) \mathbf{1}_{[i2^{-n},(i+1)2^{-n})}$$

converge pointwise to f. Note that f_n is well-defined for all $n \geq 1$ as all but one term in the sum are non-zero when applied to any $x \in \mathbb{R}$, and it is a simple function as the intervals $[i2^{-n}, (i+1)2^{-n})$ are Borel. Now for $x \in \mathbb{R}$ fixed and $\varepsilon > 0$, by continuity there is $\delta > 0$ such that $|f(x) - f(y)| \leq \varepsilon$ whenever $|x - y| \leq \delta$, so that taking n big enough (larger than $-\log_2 \delta$) one gets that

$$|f_n(x) - f(x)| = \left|f(2^{-n}\lfloor 2^n x \rfloor) - f(x)\right| \leq \varepsilon$$

which shows that $(f_n(x))_{n\geq 1}$ converges to $f(x)$, hence $(f_n)_{n\geq1}$ converges pointwise to f, which is therefore measurable.

Another proof would rely on observing the important fact that the preimage of an interval of the form (a,b) by a continuous function f is not only measurable, but actually an open set. Let (a,b) be an arbitrary open interval in $\mathbb{R}$ and suppose that $x \in f^{-1}((a,b))$; that is, $f(x) \in (a,b)$. Since (a,b) is open, there exists an $\epsilon > 0$ such that $(f(x) - \epsilon, f(x) + \epsilon) \subset (a,b)$. By continuity, there exists $\delta > 0$ such that for every $y \in \mathbb{R}$ with $|y - x| < \delta$, we have $f(y) \in (a,b)$. This shows that $B(x,\delta) \subset f^{-1}((a,b))$ and therefore that $f^{-1}((a,b))$ is open by definition: in particular, it is Borel, and we get that f is measurable. □

Exercise 5. *Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function and $f_n : \mathbb{R} \to \mathbb{R}$ its dyadic approximations given by $f_n(x) = 2^{-n}\lfloor 2^n f(x) \rfloor$. For $m \geq n \geq 1$, show the bound*

$$f(x) - 2^{-n} < f_n(x) \leq f_m(x) \leq f(x).$$

Proof. By definition

$$\lfloor 2^n f(x) \rfloor \le 2^n f(x) \quad \text{and} \quad \lfloor 2^n f(x) \rfloor > 2^n f(x) - 1.$$

So,

$$2^{-n}\lfloor 2^n f(x) \rfloor \le f(x) \quad \text{and} \quad 2^{-n}\lfloor 2^n f(x) \rfloor > f(x) - 2^{-n}.$$

from which we conclude

$$f(x) - 2^{-n} < f_n(x) \le f(x).$$

For the monotonicity in $n \in \mathbb{N}$, we take $m \ge n$. Observe that

$$2^n f(x) - 1 < \lfloor 2^n f(x) \rfloor \le 2^n f(x).$$

Multiplying by 2^{m-n} gives

$$2^{m-n}\big(2^n f(x) - 1\big) \; < \; 2^{m-n}\lfloor 2^n f(x) \rfloor \; \le \; 2^m f(x).$$

Now, since $2^{m-n}\lfloor 2^n f(x) \rfloor$ is an integer and $\lfloor 2^m f(x) \rfloor$ is the largest integer smaller or equal to $2^m f(x)$, we must have

$$2^{m-n}\lfloor 2^n f(x) \rfloor \le \lfloor 2^m f(x) \rfloor.$$

Dividing by 2^m, we get

$$2^{-n}\lfloor 2^n f(x) \rfloor \le 2^{-m}\lfloor 2^m f(x) \rfloor,$$

in other words,

$$f_n(x) \le f_m(x).$$

□

Exercise 6. *Prove Lemma 2.14 from the class, by showing first that*

$$f^{-1}([a,b)) = \bigcap_{j\ge 1} \bigcup_{k\ge 1} \bigcup_{n\ge 1} \bigcap_{m\ge n} f_m^{-1}([a - 1/j, b - 1/k))$$

Proof. A good trick to keep in mind when showing equalities of sets whenever unions and intersections appear, is to translate these operators into logical quantifiers. What we mean is the following: let us denote by A the left-hand side and B the right-hand side.

Suppose $x \in A$. Then $f(x) \in [a, b)$. By definition,

$$x \in B \iff \forall j \in \mathbb{N}, \exists\, k \in \mathbb{N}, \exists\, n \in \mathbb{N}, \forall\, m \ge n : x \in f_m^{-1}\big(\big[\, a - \tfrac{1}{j},\, b - \tfrac{1}{k}\big)\big).$$

On the other hand, we know from pointwise convergence of $(f_n)_{n\ge 1}$ to f that for all $j \ge 1$, there exists $n \in \mathbb{N}$ such that for all $m \ge n$, $|f(x) - f_m(x)| \le 1/j$. If $x \in f^{-1}([a, b))$, we can take j large enough such that $2/j < b - f(x)$, so that in fact $x \in f^{-1}([a, b - 2/j))$. But by the convergence stated above, it means that for $m \ge n$, $x \in f_m^{-1}([a - 1/j, b - 1/j))$, which shows that $x \in B$.

Conversely, let $x \in B$. Then,

$$\forall\, j \in \mathbb{N}, \exists\, k \in \mathbb{N}, \exists\, n \in \mathbb{N}, \forall\, m \ge n : f_m(x) \in \big[a - \frac{1}{j},\, b - \frac{1}{k}\big).$$

By convergence of $(f_n)_{n\ge 1}$ to f again, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m \ge N$, $|f(x) - f_m(x)| \le \varepsilon$. In particular, for $m \ge \max(n, N)$, it holds that $f(x) \in [a - \frac{1}{j} - \varepsilon, b - \frac{1}{k} + \varepsilon)$. If we choose $1/\varepsilon < \min(1/j, 1/k)$, we obtain that $a - 2/j < f(x) < b$, and since this is true for all j, it follows that $x \in A$.

□