Exercise sheet 1

An appetizer to Fourier

Exercise 1. *Consider G the graph given by a cycle of length n, i.e. G is the graph with vertices $V := \{1, \dots, n\}$ and edges $E := \{(i, i+1), i = 1, \dots, n\}$ where we identify $n+1$ to the vertex 1. Let $f : V \to \mathbb{R}$ and consider the discrete heat equation on G with initial data given by f:*

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} & = K\Delta_d u(t,x) \\ u(0,x) & = f(x) \end{cases},$$

where $K > 0$ is the diffusion constant and Δ_d the discrete Laplacian[1] on G:

$$(\Delta_d f)(x) = \frac{1}{2}(f(x+1) + f(x-1) - 2f(x)).$$

Express Δ_d as a matrix and diagonalize it explicitly (by finding the eigenvectors and eigenvalues). Explain how this gives you a solution to the heat equation with the given initial data. What about uniqueness? Check with the example of one initial heat source ($f : x \mapsto \mathbf{1}_{x=1}$) that your solution makes sense physically.

Consider now the same setup on an arbitrary homogeneous graph G, i.e. such that the number of neighbors of any vertex $x \in G$ is a constant $d \in \mathbb{N}$, and with the Laplacian given by

$$(\Delta_d f)(x) = \frac{1}{d}\sum_{y\sim x}(f(y) - f(x)),$$

where the sum is over the neighbors of x in G. Can you rigorously extend the previous argument to this more general framework?

Proof. The Laplacian can be written as a square $n \times n$ matrix:

$$\Delta = \begin{pmatrix} -1 & 1/2 & 0 & \dots & 0 & 1/2 \\ 1/2 & -1 & 1/2 & \dots & 0 & 0 \\ \dots & & & & & \\ 1/2 & 0 & \dots & 0 & 1/2 & -1 \end{pmatrix}$$

If $\omega = \exp(2\pi i/n)$ denotes the fundamental n-th root of unity, then a basis of eigenvectors of Δ is given by

$$v_k = [1 \ \omega^k \ \dots \ \omega^{k(n-1)}]$$

with associated eigenvalue

$$\lambda_k = -(1 - \cos(2\pi i/k)) = -(1 - \omega^k/2 - \omega^{-k}/2), \quad 1 \le k \le n.$$

A direct computation shows that $\langle v_i, v_j\rangle = n\delta_{ij}$: in particular, we can write $\Delta = P^* D P$, where $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, and $P = [v_1/\sqrt{n}, \dots, v_n/\sqrt{n}]$. The heat equation can then be rewritten

$$\partial_t(Pu(t)) = KD \cdot Pu, \tag{1}$$

[1]For an intuitive explanation of the definition and meaning of this operator, see `https://math.stackexchange.com/questions/50274/intuitive-interpretation-of-the-laplacian-operator`.

which admits the (unique) solution

$$(Pu)(t) = \mathrm{diag}[e^{\lambda_1 Kt} \dots e^{\lambda_n Kt}](Pu)(0), \tag{2}$$

and with the initial condition $(Pu)(0) = f$ we finally find that

$$u(t) = P^* \mathrm{diag}[e^{\lambda_1 Kt} \dots e^{\lambda_n Kt}]Pf. \tag{3}$$

In other words, f can be uniquely written as $f = \sum_{i=1}^n \alpha_i v_i/\sqrt{n}$ for some coefficients $(\alpha_i)_{i=1}^n$, and if a solution u exists at time t, it can also be written uniquely as $u(t) = \sum_{i=1}^n u_i(t)v_i/\sqrt{n}$. But since in this basis Δ is diagonal,

$$\Delta u(t) = \sum_{i=1}^n \lambda_i u_i(t) \frac{v_i}{\sqrt{n}},$$

and u being a solution together with uniqueness of the basis expansion once again requires that each coefficient $u_i(\cdot)$ solves the equation

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = \lambda_i u_i(t),$$

with unique global solution $u_i(t) = \exp(\lambda_i t)u_i(0)$. The initial condition $u(0) = f$ and uniqueness of the basis expansion once again implies that $u_i(0) = \alpha_i$, and we obtain the final solution

$$u(t) = \sum_{i=1}^n \alpha_i \exp(K\lambda_i t) \frac{v_i}{\sqrt{n}}.$$

This shows that the solution exists and is unique. Alternatively, note that Equation (2) is the unique solution to the matrix equation (1) (this can be checked for instance with the Cauchy-Lipschitz theorem, seen in Analysis II).

If $f = \delta_1$, then notice that

$$f = \sum_{i=1}^n \frac{v_i}{\sqrt{n}}$$

so that

$$u(t) = \sum_{i=1}^n \exp(K\lambda_i t) \frac{v_i}{\sqrt{n}}$$

In the case when G is an arbitrary homogeneous graph G, the Laplacian can be written as a following square $n \times n$ matrix:

$$(\Delta_d)_{ij} = \begin{cases} -1, & \text{if } i = j \\ \frac{1}{d}, & \text{if } i \sim j \\ 0, & \text{otherwise} \end{cases}$$

It's clear that Δ_d is symmetric. Let's check that it's negative definite. For all $x \in \mathbb{R}^{|G|}$,

$$x^T \Delta x = -\sum_{i \sim j} \frac{1}{d}(x_i - x_j)^2 \leq 0$$

Hence, by the spectral theorem, there exists an orthonormal eigenbasis of Δ_d, and Δ_d is diagonalizable. Namely, $\Delta = P^*DP$ with $D = \mathrm{diag}(-\lambda_1, \dots, -\lambda_n)$, and $P = [w_i, \dots, w_n]$, where $\lambda \geq 0$'s are eigenvalues and w's is an ONB of eigenvectors. Therefore, Equation (3) is the unique solution to the heat equation in this case as well. Alternatively, the solution for $f = \sum_{i=1}^n \alpha_i w_i$ can be written as

$$u(t) = \sum_{i=1}^n \alpha_i \exp(-K\lambda_i t) w_i,$$

and the derivation and proof of uniqueness can be checked replicating what is argued in the paragraph between the horizontal lines as above, replacing $v_i/\sqrt{n}$ with w_i.

□

Exercise 2 (Fourier, approximated in frequency space)**.** *Consider now the heat equation on the circle* $\mathbb{S}^1$[2]

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} &= K\Delta u(t,x) \\ u(0,x) &= f(x) \end{cases},$$

where the initial heat configuration f is given by a trigonometric polynomial[3]*:*

$$f(x) = b_0 + \sum_{n=1}^p a_n \sin(2\pi n x) + \sum_{n=1}^p b_n \cos(2\pi n x), \quad b_0, (a_n)_{n=1}^p, (b_n)_{n=1}^p \subset \mathbb{R}.$$

Prove rigorously that in this case the heat equation reduces to a finite system of ODEs. Solve them explicitly to find a solution with initial condition f (without worrying about uniqueness).

What is missing to obtain a solution for any initial $f : \mathbb{S}^1 \to \mathbb{R}$?

Proof. Suppose that at any time $t > 0$ a solution $u(t,x)$ can be written as

$$u(t,x) = b_0(t) + \sum_{n=1}^p a_n(t) \sin(2\pi n x) + \sum_{n=1}^p b_n(t) \cos(2\pi n x).$$

We can compute

$$\begin{aligned} \Delta u(x,t) &= \sum_{n=1}^p (-4\pi^2 n^2 a_n(t)) \sin(2\pi n x) + \sum_{n=1}^p (-4\pi^2 n^2 b_n(t)) \cos(2\pi n x), \\ \frac{\mathrm{d}u}{\mathrm{d}t}(t,x) &= b_0'(t) + \sum_{n=1}^p a_n'(t) \sin(2\pi n x) + \sum_{n=1}^p b_n'(t) \cos(2\pi n x). \end{aligned}$$

For the heat equation to be satisfied and the two expressions above to coincide, the coefficients behind the sine and cosine terms must agree. Indeed, for any $1 \le n \le p$, since

$$\int_0^1 \sin(2\pi n x)\sin(2\pi m x)\mathrm{d}x = \delta_{mn}/2, \int_0^1 \sin(2\pi n x)\cos(2\pi m x)\mathrm{d}x = 0, \int_0^1 \sin(2\pi n x)\mathrm{d}x = 0,$$

in order for the equality

$$\int_0^1 \Delta u(x,t)\sin(2\pi n x)\mathrm{d}x = \int_0^1 \partial_t u(x,t)\sin(2\pi n x)\mathrm{d}x$$

to be satisfied it is necessary that the coefficient $a_n(t)$ satisfy the following ODE:

$$\frac{\partial a_n(t)}{\partial t} = -4K\pi^2 n^2 a_n(t),$$

[2]Which here we understand as $[0,1]$, with all functions defined with periodic boundary conditions
[3]Equivalently, one may write $f(x) = \sum_{n=-p}^p c_n e^{inx}, (c_n)_{n=-p}^p \subset \mathbb{C}$.

and similarly

$$\frac{\partial b_n(t)}{\partial t} = -4K\pi^2 n^2 b_n(t).$$

This admits the unique global solution $a_n(t) = \exp\big(-4K\pi^2 n^2 t\big) a_n(0)$, $b_n(t) = \exp\big(-4K\pi^2 n^2 t\big) b_n(0)$. Reasoning as above (integrating against $\sin(2\pi n\cdot), \cos(2\pi n\cdot)$), we find that the coefficients $(a_n(0))_{1\leq n\leq p}, (b_n(0))_{1\leq n\leq p}$ have to match the coefficients of f. Combining all together, the final expression is:

$$u(t,x) = b_0 + \sum_{i=1}^{p} a_i \exp\big(-4K\pi^2 n^2 t\big) \sin(2\pi n x) + \sum_{i=1}^{p} b_i \exp\big(-4K\pi^2 n^2 t\big) \cos(2\pi n x).$$

For a general initial condition f, we hope to write

$$f(x) = a_0 + \sum_{i=1}^{+\infty} a_i \sin(2\pi n x) + \sum_{i=1}^{+\infty} b_i \cos(2\pi n x),$$

for some set of coefficients $b_0, (a_i)_{i=1}^p, (b_i)_{i=1}^p \subset \mathbb{R}$. If we can make sense of this mathematically, we hope to be able to replicate the steps above and write

$$u(t,x) = b_0 + \sum_{i=1}^{+\infty} a_i \exp\big(-4K\pi^2 n^2 t\big) \sin(2\pi n x) + \sum_{i=1}^{+\infty} b_i \exp\big(-4K\pi^2 n^2 t\big) \cos(2\pi n x).$$

This should be the case if $(\sin(2\pi n\cdot), \cos(2\pi n\cdot))_{n\geq 0}$ would form an orthonormal basis of some functional space in which we could take f.

□

Continuous functions

Exercise 3. *Let $D \subset \mathbb{R}^n$ be a box[4]. Find a sequence of functions $(f_k)_{k\geq 1} \subset C(D,\mathbb{R})$ that converges pointwise to a function f that is not continuous.*

Proof. One can take for instance $f_n : \mathbf{x} \mapsto \min(1, n \cdot d(\mathbf{x}, \mathbf{a}))$, where $\mathbf{a} = (a_1, \ldots, a_n)$. For each $n \geq 1$, this is a continuous function, as a minimum of continuous functions (see Exercise 6). Furthermore, we claim that $(f_n)_{n\geq 1}$ converges to $\mathbf{x} \mapsto \mathbf{1}_{\mathbf{x}\neq\mathbf{a}}$. Indeed, $f_n(\mathbf{a}) = 0$, while for $\mathbf{x} \neq \mathbf{a}$, $f_n(\mathbf{x}) = 1$ for $n > 1/d(\mathbf{x}, \mathbf{a})$. □

Exercise 4. *Find all functions $f \in C([0,1],\mathbb{R})$ such that the sequence of powers $(f^n)_{n\geq 1}$ converges in uniform norm, and characterize the possible limits.*

Proof. Let $f \in C([0,1],\mathbb{R})$ be arbitrary and $M = \sup_{x\in[0,1]} |f(x)|$, which is finite as f is continuous on a compact subset of $\mathbb{R}$, and attained at a point $y \in [0,1]$ (i.e. $M = \max_{x\in[0,1]} |f(x)| = |f(y)|$). In particular, if $M > 1$, then $(f^n(y))_{n\geq 1}$ diverges to $+\infty$ so $(f^n)_{n\geq 1}$ cannot converge in $C([0,1],\mathbb{R})$ as it doesn't even converge pointwise. If $M < 1$, then $(f^n)_{n\geq 1}$ converges to the function that is identically zero w.r.t. to the infinite norm, since $\|f^n\| \leq M^n \xrightarrow[n\to\infty]{} 0$. If $M = 1$, there are three cases: either $f \equiv 1$ in which cases the sequence of powers clearly converges to 1, either $f \equiv -1$ in which case it does not, either $|f(y)| = 1$ and $|f(z)| < 1$ for some other $z \in [0,1]$. We claim that $(f^n)_{n\geq 1}$ cannot converge in $C([0,1],\mathbb{R})$: suppose it did to a function $g \in C([0,1],\mathbb{R})$. We would have $g = \lim_{n\to\infty} f^n = \lim_{n\to\infty} f^{2n} = g^2$ so that g can only take the values $0, -1$ or 1. But on the other hand, it is clear that $g(y) = 1, g(z) = 0$. By the intermediate value theorem, g would have to take the value $1/2$ on $[y,z]$, which is the desired contradiction. This exhausts all the cases. □

Exercise 5. *Show that in fact $C(D,\mathbb{R})$ has also a multiplicative structure: if $f, g \in C(D,\mathbb{R})$, then also the product $h(x) := f(x)g(x)$ is in $C(D,\mathbb{R})$. What about the function $\max(f,g)$?*

[4] i.e. there exist $(a_i)_{i=1}^n, (b_i)_{i=1}^n, a_i < b_i \ \forall 1 \leq i \leq n$ such that $D = \prod_{i=1}^n [a_i, b_i]$.

Proof. It is clear that $h : D \to \mathbb{R}$. We show that h is continuous at $x \in D$. Define $C_f = \|f\|_\infty + 1$ and $C_g = \|g\|_\infty + 1$, which are finite since D is closed and bounded. By continuity of f and g, given $\epsilon > 0$ we can find $\delta_f > 0$ and $\delta_g > 0$ such that $|x-y| < \delta_f, |x-y| < \delta_g \implies |f(y) - f(x)| < \frac{\epsilon}{2C_f}, |g(y) - g(x)| < \frac{\epsilon}{2C_g}$. Let $\delta = \min\{\delta_f, \delta_g\}$. We have that $|x - y| < \delta$ implies

$$\begin{aligned} |f(x)g(x) - f(y)g(y)| &= |f(x)g(x) + f(x)g(y) - f(x)g(y) - f(y)g(y)| \\ &\le |f(x)g(x) - f(x)g(y)| + |f(x)g(y) - f(y)g(y)| \\ &= \underbrace{|f(x)|}_{\le C_f}\underbrace{|g(x) - g(y)|}_{< \frac{\epsilon}{2C_g}} + \underbrace{|g(y)|}_{\le C_g}\underbrace{|f(x) - f(y)|}_{< \frac{\epsilon}{2C_f}} < \epsilon. \end{aligned}$$

Now let $h = \max(f, g)$. Again it is clear that $h : D \to \mathbb{R}$. By a straightforward case analysis it is easy to see that $h = \frac{1}{2}(f + g + |f - g|)$. We show that h is continuous at $x \in D$. Given $\epsilon > 0$, by continuity of f and g we can find $\delta_f > 0$ and $\delta_g > 0$ such that $|x - y| < \delta_f, |x - y| < \delta_g \implies |f(y) - f(x)| < \frac{\epsilon}{2}, |g(y) - g(x)| < \frac{\epsilon}{2}$. For $\delta = \min\{\delta_f, \delta_g\}$ we have that $|x - y| < \delta$ implies

$$\begin{aligned} |h(x) - h(y)| &= \frac{1}{2}|f(x) + g(x) + |f(x) - g(x)| - f(y) - g(y) - |f(y) - g(y)|| \\ &\le \frac{1}{2}|f(x) - f(y)| + \frac{1}{2}|g(x) - g(y)| + \frac{1}{2}||f(x) - g(x)| - |f(y) - g(y)|| \\ &\le \frac{1}{2}|f(x) - f(y)| + \frac{1}{2}|g(x) - g(y)| + \frac{1}{2}|f(x) - f(y)| + \frac{1}{2}|g(x) - g(y)| < \epsilon, \end{aligned}$$

where in the last inequality we used the reverse triangle inequality $||a| - |b|| \le |a - b|$ and $|a - b| \le |a| + |b|$. □

Exercise 6. *Recall the axioms of a vector space and verify them in the case of* $(C(D, \mathbb{R}), +)$.

Proof.

1. *Closure under addition*: If f and g are continuous functions from D to $\mathbb{R}$, then $f + g$ is also a continuous function from D to $\mathbb{R}$.

2. *Closure under scalar multiplication*: For any scalar α and continuous function f from D to $\mathbb{R}$, the function αf is also continuous from D to $\mathbb{R}$.

3. *Commutativity of addition*: Addition of functions is commutative because addition of real numbers is commutative.

4. *Associativity of addition*: Addition of functions is associative because addition of real numbers is associative.

5. *Identity element of addition*: The constant function $0(x) = 0$ for all $x \in D$ satisfies $f + 0 = f$ for any continuous function f from D to $\mathbb{R}$. This holds because 0 acts as the additive identity for real numbers.

6. *Inverse elements of addition*: For every continuous function f from D to $\mathbb{R}$, there exists a function, denoted by $-f$, such that $f + (-f) = 0$. This function is given by $(-1) \cdot f$. This holds because $-f$ acts as the additive inverse for real numbers.

7. *Distributivity of scalar multiplication with respect to vector addition*: $\alpha(f + g) = \alpha f + \alpha g$ for any scalar α and continuous functions f and g from D to $\mathbb{R}$. This holds due to the distributive property of real numbers.

8. *Distributivity of scalar multiplication with respect to field addition*: $(\alpha + \beta)f = \alpha f + \beta f$ for any scalars α and β and continuous function f from D to $\mathbb{R}$. This also holds due to the distributive property of real numbers.

9. *Compatibility of scalar multiplication with field multiplication*: $(\alpha\beta)f = \alpha(\beta f)$ for any scalars α and β and continuous function f from D to $\mathbb{R}$. This holds due to the associative property of real numbers.

10. *Identity element of scalar multiplication*: $1f = f$ for any continuous function f from D to $\mathbb{R}$, where 1 is the multiplicative identity in the field of real numbers.

□