Exercise sheet 1

Exercise 1. *Find a sequence $(f_n)_{n\geq 1}$ of continuous functions on $[0,1]$ that converges pointwise to a function which is not continuous.*

Proof. Consider for $n \geq 1$, $f_n : x \mapsto x^n$. It is clear that this sequence of functions converges pointwise to the function $f : x \mapsto \mathbf{1}_{x=1}$, which is not continuous at 1. Note that in particular the convergence of $(f_n)_{n\geq 1}$ to f cannot be uniform, as otherwise f would be continuous by Proposition 1.3. □

Exercise 2. *Find sequences $(f_n)_{n\geq 1} \subset \mathbb{R}^{[0,1]}$ of Riemann-integrable functions converging pointwise to $f : [0,1] \to \mathbb{R}$ such that*

1. *(Limits of Riemann-integrable functions need not be Riemann-integrable) f is not Riemann integrable.*
2. *(Limits of integrals doesn't equal the integral of the limit) f is Riemann integrable, but $\int_0^1 f_n(x)\mathrm{d}x$ does not converge to $\int_0^1 f(x)\mathrm{d}x$.*

Proof.

1. Consider $\{q_n\}_{n\geq 1} = \mathbb{Q}\cap[0,1]$ an enumeration of the rationals and $f_n = \mathbf{1}_{\{q_1,\ldots,q_n\}}$. For each fixed $n \geq 1$, the function f_n is Riemann integrable as it is continuous except on the finite number of points $\{q_1,\ldots,q_n\}$ (we refer to Analysis I for that result). However, $f_n \xrightarrow[n\to\infty]{} f = \mathbf{1}_{\mathbb{Q}}$, which is not Riemann-integrable. Indeed, for each partition $P = \{a_0, a_1, \ldots, a_n\}$ of $[0,1]$, by density we have that $U(P,f) = 1, L(P,f) = 0$, so Riemann's criterion (Theorem 8 in the reminders sheet) cannot be verified for $\varepsilon < 1$.
2. Define for $n \geq 1$ the function

$$f_n : x \mapsto \begin{cases} 4n^2 x & : 0 \leq x \leq \frac{1}{2n} \\ 4n - 4n^2 x & : \frac{1}{2n} \leq x \leq \frac{1}{n} \\ 0 & : \frac{1}{n} \leq x \leq 1. \end{cases}$$

It is straighforward to check that it is integrable with $\int_0^1 f_n = 1$, but we also have that $f_n \xrightarrow[n\to\infty]{} f \equiv 0$ pointwise, which is Riemann integrable with integral 0.

□

Exercise 3 (Sequences, I)**.** *Prove the Bolzano-Weierstrass theorem: for any $N \geq 1$ and $K \subset \mathbb{R}^N$ closed and bounded, every sequence $(x_n)_{n\geq 1} \subset K$ admits a subsequence (x_{n_k}) that converges in K.*

Proof. Let us consider $(x_n)_{n\geq 1} \subset K$ as in the statement and construct a convergent subsequence. For all $k \geq 0$, one can write $K = \bigcup_{\mathbf{i}\in\Lambda_k} K \cap C^k_{\mathbf{i}}$, where $C^k_{\mathbf{i}}$ is a cube of dimensions 2^{-k} with center $2^{-k}\mathbf{i}$ and with edges parallel to the axes, and $\Lambda_k = \left\{\mathbf{i} \in \mathbb{Z}^n : K \cap C^k_{\mathbf{i}} \neq \emptyset\right\}$ is finite since K is bounded. In particular for $k = 1$, because there are only finitely many cubes with $\mathbf{i} \in \Lambda_1$, there must exist $\mathbf{i}_1 \in \Lambda_1$ such that $K \cap C^k_{\mathbf{i}_1}$ contains infinitely many terms of $(x_n)_{n\geq 1}$: take an arbitrary one x_{i_1}. We can now repeat the argument inductively: writing $K \cap C^1_{\mathbf{i}_1} = \bigcup_{\mathbf{i}\in\Lambda_2}(K \cap C^1_{\mathbf{i}_1}) \cap C^2_{\mathbf{i}}$, there is a cube $C^2_{\mathbf{i}_2}$ that contains infinitely many terms of $(x_n)_{n\geq 1}$, so that we can choose any term x_{i_2} with $i_2 > i_1$.

Doing so we obtain a subsequence $(x_{i_k})_{k\geq 1}$, that is Cauchy by construction: indeed for $\varepsilon > 0$, picking M such that $2^{-M} < \varepsilon/\sqrt{n}$, we see that for all $k_1, k_2 \geq M$, $x_{i_{k_1}}$ and $x_{i_{k_2}}$ belong to $C^M_{\mathbf{i}_M}$, so that $\|x_{i_{k_1}} - x_{i_{k_2}}\| < \sqrt{n}2^{-M} < \varepsilon$. By completeness of $(\mathbb{R}^n, \|\cdot\|)$, there exists $x \in \mathbb{R}^n$ such that $(x_{i_k})_{k\geq 1}$ converges to x. But since K is closed, the limit x must also be in K, and this concludes the proof. □

Exercise 4 (Sequences, II)**.** *Let $K \subset \mathbb{R}^N$ be closed and bounded and $f \in C(K, \mathbb{R})$. Prove that there exist $\underline{x}, \overline{x} \in K$ such that*

$$f(\underline{x}) = \inf_{x\in K} f(x), \quad f(\overline{x}) = \sup_{x\in K} f(x).$$

Proof. Consider $M = \sup_{x\in K} f(x)$ and a sequence $(x_n)_{n\geq 1}$ such that $f(x_n) \xrightarrow[n\to\infty]{} M$ by definition of the supremum. Because K is compact, there exists $\overline{x} \in K$ and a subsequence $(x_{n_k})_{k\geq 1}$ such that $x_{n_k} \xrightarrow[k\to\infty]{} \overline{x}$. By continuity of f at $\overline{x}$, we must have that $f(x_{n_k}) \xrightarrow[k\to\infty]{} f(\overline{x})$, enforcing $f(\overline{x}) = M$.

For counterexamples, one can consider $f : x \mapsto \|x\|$ when K is not bounded, where $\sup_{x\in K} f(x) = +\infty$. When K is not closed, $\mathbb{R}^n \setminus K$ is not open, so there is a point $x \in \mathbb{R}^n \setminus K$ such that for all $\varepsilon > 0$, $B(x, \varepsilon) \cap K \neq \emptyset$. We can define the function $f : y \mapsto 1/d(x, y)$: it is continuous on K but unbounded near x, as for $y \in B(x, \varepsilon) \cap K$ we have $f(y) \geq 1/\varepsilon$, for arbitrary $\varepsilon > 0$. □

Exercise 5. *In this exercise, we try to generalise the definition of sums that you have already seen:*

1. *Recall the definition of a converging and absolutely converging sequence, and the relevant results.*

2. *Consider a convergent series $\sum_{n=1}^{+\infty} a_n$, i.e. such that $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$ exists. Is it true that*

$$\sum_{n=1}^{+\infty} a_n = \sum_{n=1}^{+\infty} a_{2n} + \sum_{n=1}^{+\infty} a_{2n-1},$$

   i.e. can you sum the even and odd terms separately? (this writing implicitly requires to check if the series converge) If not, what condition can you add to make it true?

3. *Suppose now that for each $n \in \mathbb{Z}$ we have some number $a_n \in \mathbb{R}$, and we want to define something like*

$$\sum_{n\in\mathbb{Z}} a_n.$$

   How would you make sense of this? And if you can think of different ways to define it, can you find sufficient conditions so that they agree?

4. *In particular, is it true that if $\sum_{n\in\mathbb{Z}} a_n$ converges, then $\sum_{n=1}^{+\infty} a_n$ (and $\sum_{n=1}^{+\infty} a_{-n}$) does too, and*

$$\sum_{n\in\mathbb{Z}} a_n = a_0 + \sum_{n=1}^{+\infty} a_n + \sum_{n=1}^{+\infty} a_{-n}.$$

Proof. 2. In general, such an identity does not hold: for instance, the series of general term $a_n = (-1)^n/n$ converges (by the alternated series criterion, see Analysis I), but $\sum_{n=1}^{+\infty} a_{2n}$ corresponds to the harmonic series and diverges to $+\infty$ (similarly, $\sum_{n=1}^{+\infty} a_{2n-1} = -\infty$). On the other hand, if the series $\sum_{n=1}^{+\infty} a_n$ converges absolutely, i.e. if $\sum_{n=1}^{+\infty} |a_n|$ converges, then it is correct. Indeed, we first note that by definition of absolute convergence, for all $\varepsilon > 0$, there exists $N > 0$ such that for all $n, m \geq N$,

$$\left| \sum_{k=1}^{n} |a_k| - \sum_{j=1}^{m} |a_j| \right| \leq \varepsilon$$

(since $\sum_{k=1}^{n} |a_n|$ converges, it is Cauchy), so that in particular taking $m = N$ it holds that

$$\sum_{k=N}^{n} |a_k| \leq \varepsilon.$$

Then, for all $n, m \geq N/2$,

$$\left| \sum_{k=1}^{n} a_{2k} - \sum_{j=1}^{m} a_{2j} \right| \leq \sum_{k=N}^{\max(2n,2m+1)} |a_{2k}| \leq \varepsilon,$$

which means that $(\sum_{k=1}^{n} a_{2k})_{n\geq 1}$ is Cauchy and therefore converges. Similarly, we argue that $(\sum_{k=1}^{n} a_{2k+1})_{n\geq 1}$ converges. To prove the identity it is now sufficient to see that[1]

$$\left| \sum_{n=1}^{N} a_n - \sum_{n=1}^{N} a_{2n} - \sum_{n=1}^{N} a_{2n-1} \right| \leq \sum_{n=N+1}^{2N} |a_n| \leq \varepsilon.$$

3-4. It is natural to define $\sum_{n\in\mathbb{Z}} a_n$ as the limit of the sequence $(\sum_{n=-N}^{+N} a_n)_{N\geq 1}$, if it exists. However, this definition suffers from the same problems as 'simple' convergence of a series (compared to absolute convergence), in that it doesn't imply convergence of the series in a different summation order. For this reason, it is sometimes called the 'principal value' of the series $\sum_{n\in\mathbb{Z}} a_n$. For instance, if $a_n = \text{sign}(n)$ (with the convention $\text{sign}(0) = 0$), then with the definition above $\sum_{n\in\mathbb{Z}} a_n$ exists and is equal to 0. On the other hand, neither $\sum_{n=1}^{+\infty} a_n$ nor $\sum_{n=1}^{+\infty} a_{-n}$ exist. To obtain a better-behaved definition, we must ask that $\sum_{n\in\mathbb{Z}} a_n$ converges absolutely, i.e. that $(\sum_{n=-N}^{+N} |a_n|)_{N\geq 1}$ converges. This implies again that $(\sum_{n=-N}^{+N} a_n)_{N\geq 1}$ converges (check it!), so that this notion of convergence is stronger than the previous one, and the proof of the identity in the statement is almost identical to that of item 2: for all $\varepsilon > 0$ there exists $N > 0$ such that for all $n \geq N$,

$$\sum_{N\leq |k|\leq n} |a_k| \leq \varepsilon,$$

implying that for all $n, m \geq N$,

$$\left| \sum_{k=1}^{n} a_k - \sum_{j=1}^{m} a_j \right| \leq \sum_{k=N}^{\max(n,m)} |a_k| \leq \varepsilon,$$

$$\left| \sum_{k=1}^{n} a_{-k} - \sum_{j=1}^{m} a_{-j} \right| \leq \sum_{k=N}^{\max(n,m)} |a_{-k}| \leq \varepsilon,$$

so that both $\sum_{n=1}^{+\infty} a_n$ and $\sum_{n=1}^{+\infty} a_{-n}$ converge (actually absolutely). Then,

$$\left| \sum_{n=-N}^{N} a_n - \sum_{n=1}^{N} a_n - \sum_{n=1}^{N} a_{-n} - a_0 \right| = 0 \leq \varepsilon$$

achieves to prove the identity.

$\square$

[1] For $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}, (c_n)_{n\geq 1}$ sequences of real numbers, to prove that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n + \lim_{n\to\infty} c_n$$

it is enough to prove that the limits exist (otherwise the statement is vacuous!) and that for all $\varepsilon > 0$, there exists $N \geq 1$ such that for all $n \geq N$, $|a_n - b_n - c_n| \leq \varepsilon$.