# Exercise sheet 9

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

**Exercise 1.** *Recall that a finite simple function is a function $f : \mathbb{R}^n \to \mathbb{R}$ that may be written*

$$f = \sum_{i=1}^{n} c_i \mathbf{1}_{E_i}$$

*where $n \geq 1$, $(c_i)_{i=1}^n \subset \mathbb{R}$ and $(E_i)_{i=1}^n \subset \mathbb{R}^n$ are disjoint Borel sets.*

*Show that for any measurable function $g : \mathbb{R}^n \to \mathbb{R}$, there exists a sequence of finite simple functions converging pointwise to $g$.*

*Proof.* We begin first with a non-negative measurable function $g$. The general case is obtained by applying the following approximation on the positive and negative part of $g$.

To construct the sequence $(f_n)_{n\geq 1}$ approximating $g$, we define the sets $E_{k,n} \subset \mathbb{R}^n$ by

$$E_{k,n} = g^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right)$$

which are measurable since $g$ is. We then define $f_n$ by

$$f_n(x) \ = \ \sum_{k=1}^{n2^n} \frac{k}{2^n} \mathbb{1}_{E_{k,n}}$$

By construction, we have $f_n \leq f_{n+1} \leq g(x)$ and for all $n > g(x)$, $0 \leq g(x) - f_n(x) \leq 2^{-n}$. Therefore, $f_n(x) \to g(x)$. □

**Exercise 2.** *Show that $x \mapsto \exp(-|x|)$ is integrable over $\mathbb{R}$, but the constant function $c$ for example is not integrable for $c \neq 0$. Is $f : x \mapsto x$ integrable over $\mathbb{R}$?*

*Proof.* First, note that the function $f : x \mapsto \exp(-|x|)$ is continuous and therefore measurable. Moreover, it is non-negative, so to check integrability we only need to show that its integral is finite.

Consider the simple functions

$$g_n(x) := \sum_{k=0}^{\infty} \exp\left(-\frac{k}{n}\right) \mathbb{1}_{[k/n,(k+1)/n)}(x)$$

for $x \geq 0$, and observe that the sequence $(g_n)$ is increasing with $g_n(x) \geq \exp(-x)$ for $x \geq 0$. These essentially approximate $\exp(-|\cdot|)$ from above over the positive real line. Observe that they are integrable, as

$$\sum_{k=0}^{\infty} \exp\left(-\frac{k}{n}\right) \lambda[k/n, (k+1)/n) \leq \int_{-1}^{\infty} \exp(-x) dx = e < \infty$$

where the integral here denotes the Riemann integral. Therefore, by monotonicity of the Lebesgue integral, it follows that $f\mathbb{1}_{\mathbb{R}_+}$ is integrable, and by additivity that $f$ is integrable (since the same argument shows that $f\mathbb{1}_{\mathbb{R}_-}$ is integrable). We also have in particular that

$$\int_{\mathbb{R}} \exp(-|x|) \mathrm{d}\lambda \leq 2e < \infty.$$

Next, we consider the constant function $g : x \mapsto c$ with $c \neq 0$, which is itself a simple function. Over all of $\mathbb{R}$, the Lebesgue measure is infinite. Hence,

$$\int_{\mathbb{R}^n} c\,\lambda(\mathrm{d}x) := \sup\left\{\int_{\mathbb{R}^n} g(x)\,\lambda(\mathrm{d}x)\,\middle|\, 0 \leq g \leq c,\ g \text{ simple}\right\}$$
$$\geq \sup_{n\geq 1} \int c\mathbb{1}_{[-n,n]}(x)\lambda(\mathrm{d}x) = \sup_{n\geq 1} 2cn = +\infty.$$

Thus, the constant function $c \neq 0$ is not integrable[1] over $\mathbb{R}$.

Similarly,

$$\int_{\mathbb{R}^n} |x|\,\lambda(\mathrm{d}x) := \sup\left\{\int_{\mathbb{R}^n} g(x)\,\lambda(\mathrm{d}x)\,\middle|\, 0 \leq g(x) \leq |x|\ \forall x \in \mathbb{R},\ g \text{ simple}\right\}$$
$$\geq \sup_{n\geq 1} \int \mathbb{1}_{[1,n]}(x)\lambda(\mathrm{d}x) = \sup_{n\geq 0} n = +\infty.$$

so that $f : x \mapsto x$ is not integrable (since $|f|$ is not). □

**Exercise 3.** *Let $f, g : \mathbb{R} \to \mathbb{R}$ be measurable and integrable. Is $fg$ necessarily integrable? What if $|g(x)| \leq 1$ for all $x \in \mathbb{R}$?*

*Proof.* That is not true in general: let $f, h$ equal the first function in the correction of Exercise 4, sheet 8 (denoted $f$ there). Then both are integrable, but their product $fh$ is the function $g$ of the correction, which is not integrable.[2] If $|h(x)| \leq 1$ for all $x \in \mathbb{R}$, then $|fh| \leq |f|$ from which it follows by monotonicity of the integral that $|fh|$ is integrable, and so is $fh$ as well. □

**Exercise 4.** *In probability/statistics you have seen the notion of a random variable $X$. Suppose $X$ takes values in the set $C = \{c_1, c_2, \dots\}$ each value with probability $\mathbb{P}(c_i)$. What is the expectation (or mean value) of this random variable? Compare this to the Lebesgue integral of a simple function that takes values in the set $C$.*

*Proof.* The random variable $X$ is a measurable function $\Omega \to C$, for some probability space $\Omega$. When $C$ is countable, the expectation is defined as

$$\sum_{n=1}^{+\infty} c_n \mathbb{P}[X = c_n].$$

More generally, this is actually equal to the integral

$$\int_\Omega X \mathrm{d}\mathbb{P},$$

in the sense of Lebesgue. This is akin to the definition of the integral of a simple function. □

**Exercise 5.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be integrable and $a \in \mathbb{R}$. Verify that $af$ is integrable and show that $\int af d\lambda = a \int f d\lambda$ (homogeneity).*

The mapping $x \mapsto a$ is measurable, and so is $f$ by the integrability assumption. Therefore, $af$ is measurable.

Note we have already shown the homogeneity property claimed for simple functions in sheet 8, exercise 6 (set $g = 0$ or $b = 0$). Take a non-negative integrable function $f$ and $a \geq 0$. In particular,

[1] In fact, the definition of the integral of a simple function readily implies that it is not integrable. We detail the argument here a bit differently, to avoid confusion.

[2] This essentially amounts to taking $f, h : x \mapsto 1/\sqrt{x}$, except that choosing simple functions from the start allow for an immediate computation of the integral.

$f$ is measurable. By exercise 1, we can approximate it from below using a sequence $(f_n)$ of finite simple functions. Observe that $(af_n)$ approximates $af$ from below. Thus, using the monotone convergence theorem (MCT), we get that $af$ is integrable with

$$\int af \mathrm{d}\lambda = \lim_n \int af_n \mathrm{d}\lambda = a \lim_n \int f_n \mathrm{d}\lambda = a \int f \mathrm{d}\lambda$$

where we used sheet 8, exercise 6 in the second equality.

To see that homogeneity of the integral also holds for any integrable $f$ and $a \in \mathbb{R}$, we split $af = (af)_+ - (af)_-$, where $(af)_\pm$ denotes the positive and negative part of $af$, respectively. Noting that both $(af)_\pm \geq 0$ are non-negative and integrable, the claim follows from the previous step.

Remark: The above procedure is fairly standard: One shows an integral property for (often non-negative) finite simple functions, and extends it to a larger class of functions using an approximation argument (here: exercise 1) together with an integral convergence theorem (here: MCT).

Remark: in fact here the definition of the Lebesgue integral with the supremum allows for a direct and easier proof.

**Exercise 6.** *Let $f_1 \geq f_2 \geq \dots$ be a decreasing sequence of integrable functions, converging point-wise to an integrable function $f$. Show that $\int f_n \mathrm{d}\lambda \xrightarrow[n\to\infty]{} \int f \mathrm{d}\lambda$.*

*Proof.* The idea is to use the MCT. Define

$$g_n = f_1 - f_n.$$

Since $f_1 \geq f_n$, we have that $g_n \geq 0$ for all $n$. Moreover, as $(f_n)$ is decreasing in $n$,

$$g_n = f_1 - f_n \leq f_1 - f_{n+1} = g_{n+1},$$

the sequence $(g_n)$ is increasing and

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} (f_1 - f_n) = f_1 - f,$$

which is integrable since $f_1$ and $f$ are, and a linear combination of integrable functions remains integrable.

Therefore, $(g_n)$ satisfies the assumptions of the MCT and hence, together with the linearity of the integral:

$$\begin{aligned} \int f_1 \, d\lambda - \int f \, d\lambda &= \lim_{n\to\infty} \Big( \int f_1 \, d\lambda - \int f_n \, d\lambda \Big) \\ &= \lim_{n\to\infty} \int g_n \, d\lambda = \int \lim_{n\to\infty} g_n \, d\lambda = \int (f_1 - f) \, d\lambda. \end{aligned}$$

Subtracting $\int f_1 d\lambda \in \mathbb{R}$ on both sides and using linearity again yields the desired claim. □