Exercise sheet 8

Disclaimer: the exercises are arranged by theme, not by order of difficulty.

Measurability

Exercise 1. *Prove that if $f, g : \mathbb{R}^n \to \mathbb{R}$ are Borel measurable then $\{x \in \mathbb{R}^n : f(x) = g(x)\}$ and $\{x \in \mathbb{R}^n : f(x) \neq g(x)\}$ are Borel measurable.*

Proof. The set $\{x \in \mathbb{R}^n : f(x) \neq g(x)\}$ might be rewritten $(f-g)^{-1}(\mathbb{R} \setminus \{0\})$. By exercise 3 of sheet 7, $f - g$ is measurable, and by exercise 2 of the same sheet we have that

$$(f-g)^{-1}(\mathbb{R} \setminus \{0\}) = ((f-g)^{-1}(-\infty, 0)) \cup ((f-g)^{-1}(0, +\infty))$$

is Borel. Since finally

$$\{x \in \mathbb{R}^n : f(x) = g(x)\} = \{x \in \mathbb{R}^n : f(x) \neq g(x)\}^c,$$

we conclude.

Note that the exercise can also be proven in a more basic though lengthier way, writing

$$\begin{aligned}\{x : f(x) \neq g(x)\} &= \{x : f(x) < g(x)\} \cup \{x : f(x) > g(x)\} \\ &= \bigcup_{q \in \mathbb{Q}} \{x : f(x) < q < g(x)\} \cup \{x : g(x) < q < f(x)\} \\ &= \bigcup_{q,l,r \in \mathbb{Q}} (\{x : l \leq f(x) < q\} \cap \{x : q \leq g(x) < r\}) \cup (\{x : l \leq g(x) < q\} \cap \{x : q \leq f(x) < r\}) \\ &= \bigcup_{q,l,r \in \mathbb{Q}} (f^{-1}([l,q)) \cap g^{-1}([q,r))) \cup (g^{-1}([l,q)) \cap f^{-1}([q,r))).\end{aligned}$$

By measurability of f and g all sets appearing in the last term are measurable, and since the unions and intersections are countable it follows that $\{x : f(x) \neq g(x)\}$ is measurable too. Finally since $\{x : f(x) = g(x)\} = \{x : f(x) \neq g(x)\}^c$, it is measurable as well. □

Exercise 2. *Let $(f_n)_{n \geq 1}$ be a sequence of measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$. Prove that $\sup_n f_n(x)$ and $\inf_n f_n(x)$ are also measurable.*

Proof. It is easy to check that

$$\{x : a \leq \inf_n f_n(x) < b\} = \{x : \forall n \geq 1, f_n(x) \geq a \text{ and } \exists m \geq 1, f_m(x) < b\}.$$

So, for $f = \inf_n f_n$ we have[1]

$$f^{-1}([a,b)) = \bigcap_{n \geq 1} \left(\bigcup_{q \in \mathbb{Q}} f_n^{-1}([a,q)) \right) \cap \bigcup_{m \geq 1} f_m([a,b)),$$

[1] More directly, one can use Exercise 7.2 and see that

$$f^{-1}(-\infty, b)) = \bigcup_{n \geq 1} f_n^{-1}((-\infty, b)).$$

which is a countable union of measurable sets as $(f_n)_{n\geq 1}$ is a sequence of measurable functions, and hence f is measurable. Using that $\inf_n f_n$ is measurable we now show that $\sup_n f_n$ is also measurable. Since for any set $S \subseteq \mathbb{R}$ we have $\sup_{y\in S} y = -\inf_{y\in S} -y$, we have for any $x \in \mathbb{R}$ that $\sup_n f_n(x) = -\inf_n -f_n(x)$, or in other words $\sup_n f_n = -\inf_n -f_n$. We conclude by using the fact that if g is measurable, then $-g$ is measurable. □

Lebesgue integral

Exercise 3. *Prove that the Lebesgue measure of the Cantor set is zero.*

Proof. Recall that we can write the cantor set as $C = \bigcap_{n\geq 1} C_n$, where each C_n contains 2^n closed and disjoint intervals of length $(1/3)^n$. This implies that, for any $n \in \mathbb{N}$

$$\lambda(C) = \lambda(\cap_{m\geq 1} C_m) \leq \lambda(C_n) = \left(\frac{2}{3}\right)^n,$$

which can be made arbitrarily small. □

Exercise 4 (Examples)**.** *Find a measurable function $f : \mathbb{R} \to \mathbb{R}$ that is unbounded, but is still integrable. Find also a measurable function f that is non-zero only on $[0,1]$ but is still not integrable.*

Proof. Consider the following function:

$$f(x) = \sum_{n=1}^{+\infty} 2^{n/2} \mathbf{1}_{(2^{-n}, 2^{-n+1}]}(x).$$

Being a simple function, it is measurable, and also clearly unbounded. However, since

$$\sum_{n=1}^{+\infty} 2^{n/2} \lambda((2^{-n}, 2^{-n+1}]) = \sum_{n=1}^{+\infty} 2^{n/2} 2^{-n} < +\infty,$$

it is integrable.

On the other hand, for

$$g(x) = \sum_{n=1}^{+\infty} 2^{n} \mathbf{1}_{(2^{-n}, 2^{-n+1}]}(x),$$

we have that

$$\sum_{n=1}^{+\infty} 2^{n} \lambda((2^{-n}, 2^{-n+1}]) = \sum_{n=1}^{+\infty} 2^{n} 2^{-n} = +\infty,$$

so that g is not integrable.

□

Exercise 5 (Basic properties of the Lebesgue integral)**.** *Show Lemma 2.26 from the class, i.e. that for $f : \mathbb{R}^n \to \mathbb{R}$ measurable:*

2. *if $|f(x)| \leq C$ for all $x \in \mathbb{R}^n$, then it is integrable over any finite box $[a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n$*
3. *if $\lambda(f \neq 0) := \lambda(\{x : f(x) \neq 0\}) = 0$, then f is integrable and $\int f \mathrm{d}\lambda = 0$*
4. *if $f \geq 0$ and $\int f \mathrm{d}\lambda = 0$, then $\lambda(f \neq 0) = 0$.*

Proof. Recall that a non-negative function f is said to be integrable if

$$\int_{\mathbb{R}^n} f(x)\, \lambda(\mathrm{d}x) := \sup \left\{ \int_{\mathbb{R}^n} g(x)\, \lambda(\mathrm{d}x) \,\middle|\, 0 \leq g \leq f,\ g \text{ simple} \right\} < \infty.$$

Note that in particular, if $0 \le f_1 \le f_2$ are integrable, then necessarily by monotonicity of the supremum

$$\begin{aligned}\int_{\mathbb{R}^n} f_1(x)\,\lambda(\mathrm{d}x) &= \sup\left\{\int_{\mathbb{R}^n} g(x)\,\lambda(\mathrm{d}x)\,\middle|\, 0 \le g \le f_1,\ g \text{ simple}\right\}\\ &\le \sup\left\{\int_{\mathbb{R}^n} g(x)\,\lambda(\mathrm{d}x)\,\middle|\, 0 \le g \le f_2,\ g \text{ simple}\right\} = \int_{\mathbb{R}^n} f_2(x)\,\lambda(\mathrm{d}x),\end{aligned}$$

which is often referred to as monotonicity of the (Lebesgue) integral. Furthermore, recall that a general (not necessarily non-negative) function f is integrable if f_+ and f_- are, and then

$$\int_{\mathbb{R}^n} f\mathrm{d}\lambda := \int_{\mathbb{R}^n} f_+\mathrm{d}\lambda - \int_{\mathbb{R}^n} f_-\mathrm{d}\lambda.$$

An equivalent condition for f to be integrable is then that $|f|$ is integrable, as since $|f| = f_+ + f_-$ we have that $|f|$ is integrable if and only if f_+ and f_- are (and that ultimately $\int_{\mathbb{R}^n} f\mathrm{d}\lambda \le \int_{\mathbb{R}^n} |f|\mathrm{d}\lambda$ by monotonicity). This is important in practice as to check that a function is integrable, it boils down to checking whether its absolute value–which is non-negative–is integrable. Also, if the integral of $|f|$ is zero, then so is that of f.

2. Let $B := [a_1, b_1] \times \cdots \times [a_n, b_n]$. Then, $|f(x)| \le C$ on B, i.e. $|f|\mathbf{1}_B \le C\mathbf{1}_B$. The latter is a simple function, integrable since $\lambda(B) < +\infty$ by assumption, so that

   $$\int_{\mathbb{R}^n} C\mathbf{1}_B\mathrm{d}\lambda = C \cdot \lambda(B) < +\infty.$$

   It follows from monotonicity that $|f|\mathbf{1}_B$ is integrable, i.e.

   $$\int_{\mathbb{R}^n} |f|\mathbf{1}_B\,\mathrm{d}\lambda \le \int_{\mathbb{R}^n} C\mathbf{1}_B\,\mathrm{d}\lambda = C \cdot \lambda(B) < \infty,$$

   and therefore f is integrable as well.

3. Let $A := \{x \in \mathbb{R}^n : f(x) \ne 0\}$. Note that this set is measurable due to exercise 1. By assumption, $\lambda(A) = 0$. Then, for any simple function $0 \le g \le |f|$, we have that $g = 0$ almost everywhere. Indeed, $g(x) > 0$ implies $|f(x)| > 0$, so that $\{g > 0\} \subset A$, and thus $0 \le \lambda(\{g > 0\}) \le \lambda(A) = 0$. Writing $g = \sum_{i=1}^N c_i\mathbf{1}_{E_i}$ where $(c_i)_{i=1}^{n_f} \subset \mathbb{R}$ and $(E_i)_{i=1}^N$ are disjoint Borel sets, this implies that $\lambda(E_i) = 0$ whenever $c_i \ne 0$. In particular, $\sum_{i=1}^N |c_i|\lambda(E_i) = 0$ so that g is integrable with $\int g\mathrm{d}\lambda = 0$. Finally,

   $$\int_{\mathbb{R}} |f|\mathrm{d}\lambda := \sup\left\{\int g\ \mathrm{d}\lambda \mid 0 \le g \le |f| \text{ simple}\right\} = 0.$$

   It then follows that f is integrable with integral zero.

4. Suppose for the sake of contradiction that

   $$\lambda(\{f > 0\}) > 0.$$

   Consider for $\varepsilon > 0$ the level set

   $$A_\epsilon := \{x \in \mathbb{R}^n : f(x) \ge \epsilon\}$$

   which is measurable, since f is. We notice that

   $$\{x \in \mathbb{R}^n : f(x) > 0\} = \bigcup_{n\ge1} A_{1/n},$$

and from Proposition 2.6 (2)

$$\lim_{n\to\infty} \lambda(A_{1/n}) = \lambda(\{x \in \mathbb{R}^n : f(x) > 0\}) > 0.$$

In particular, there must exist n large enough so that $\lambda(A_{1/n}) > 0$. Let us denote $\varepsilon = 1/n$. Now, $f \cdot \mathbf{1}_{A_\epsilon}$ is measurable as a product of measurable functions, and since on A_ε it holds that $f(x) \geq \varepsilon$, we obtain that

$$f(x) \cdot \mathbf{1}_{A_\epsilon}(x) \geq \epsilon \cdot \mathbf{1}_{A_\epsilon}(x) \quad \forall x \in \mathbb{R}^n.$$

The function $\epsilon \cdot \mathbf{1}_{A_\epsilon}$ is simple and non-negative, and integrable as smaller than f which is integrable by assumption. We can further compute its integral explicitly as

$$\int_{\mathbb{R}^n} \epsilon \cdot \mathbf{1}_{A_\epsilon}(x)\, \mathrm{d}\lambda = \epsilon \cdot \lambda(A_\epsilon) > 0.$$

Since $f \geq f \cdot \mathbf{1}_{A_\epsilon} \geq \epsilon \cdot \mathbf{1}_{A_\epsilon}$, we conclude by monotonicity of the Lebesgue integral that:

$$\int_{\mathbb{R}^n} f\, \mathrm{d}\lambda \geq \int_{\mathbb{R}^n} f \cdot \mathbf{1}_{A_\epsilon}\, \mathrm{d}\lambda \geq \epsilon \cdot \lambda(A_\epsilon) > 0,$$

which is the desired contradiction.

□

Exercise 6 (Linearity of the integral for simple functions)**.** *Prove that for $f, g : \mathbb{R}^n \to \mathbb{R}$ finite simple functions and $a, b \in \mathbb{R}$, $af + bg$ is integrable and*

$$\int (af + bg)\mathrm{d}\lambda = a \int f\, \mathrm{d}\lambda + b \int g\, \mathrm{d}\lambda.$$

Proof. Let $f, g \geq 0$ be finite simple functions, admitting the expression

$$f = \sum_{i=1}^{n_f} \alpha_i \mathbf{1}_{F_i}, \quad g = \sum_{j=1}^{n_g} \beta_j \mathbf{1}_{G_j},$$

with $(\alpha_i)_{i=1}^{n_f}, (\beta_j)_{j=1}^{n_g} \subset \mathbb{R}$ and $(F_i)_{i=1}^{n_f}$ (resp. $(G_j)_{j=1}^{n_g}$) disjoint Borel sets in $\mathbb{R}^n$. It follows analogously to Exercise 3, sheet 7 that $af + bg$ is also simple and can be written

$$af + bg = \sum_{i=1}^{n_f} \sum_{j=1}^{n_g} (a\alpha_i + b\beta_j) \mathbf{1}_{F_i \cap G_j}.$$

We have that

$$\begin{aligned}\sum_{i=1}^{n_f} \sum_{j=1}^{n_g} |a\alpha_i + b\beta_j| \lambda(F_i \cap G_j) &\leq \sum_{i=1}^{n_f} \sum_{j=1}^{n_g} (|a\alpha_i| + |b\beta_j|) \lambda(F_i \cap G_j) \\ &= |a| \sum_{i=1}^{n_f} |\alpha_i| \lambda(F_i) + |b| \sum_{j=1}^{n_g} |\beta_j| \lambda(G_j) < +\infty,\end{aligned}$$

implying that $af + bg$ is integrable, and

$$\begin{aligned}\int (af + bg)\mathrm{d}\lambda &= \sum_{i=1}^{n_f} \sum_{j=1}^{n_g} (a\alpha_i + b\beta_j) \lambda(F_i \cap G_j) \\ &= a \sum_{i=1}^{n_f} \alpha_i \lambda(F_i) + b \sum_{j=1}^{n_g} \beta_j \lambda(G_j) \\ &= a \int f \mathrm{d}\lambda + b \int g \mathrm{d}\lambda,\end{aligned}$$

which is the desired identity.

□